

# Neighborhoods at infinity and the Plancherel formula for a reductive $p$ -adic symmetric space

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abbreviated title: Plancherel formula for a reductive  $p$ -adic symmetric  
space

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**Abstract** Yiannis Sakellaridis and Akshay Venkatesh have determined, when the group  $G$  is split and the field  $\mathbf{F}$  is of characteristic zero, the Plancherel formula for any spherical space  $X$  for  $G$  modulo the knowledge of the discrete spectrum.

The starting point is the determination of good neighborhoods at infinity of  $X/J$ , where  $J$  is a small compact open subgroup of  $G$ . These neighborhoods are related to "boundary degenerations" of  $X$ . The proof of their existence is made by using wonderful compactifications.

In this article we will show the existence of such neighborhoods assuming that  $\mathbf{F}$  is of characteristic different from 2 and  $X$  is symmetric. In particular, one does not assume that  $G$  is split. Our main tools are the Cartan decomposition of Benoist and Oh, our previous definition of the constant term and asymptotic properties of Eisenstein integrals due to Nathalie Lagier .

Once the existence of these neighborhoods at infinity of  $X$  is established, the analog of the work of Sakellaridis and Venkatesh is straightforward and leads to the Plancherel formula for  $X$ .

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# 1 Introduction

Let  $G$  be the group of  $\mathbf{F}$ -points of a reductive group  $\underline{G}$  defined over the non archimedean local field  $\mathbf{F}$ .

In a tremendous work (cf. [SV]), Yiannis Sakellaridis and Akshay Venkatesh have determined, when the group  $G$  is split and the field  $\mathbf{F}$  is of characteristic zero, the Plancherel formula for any spherical space  $X$  for  $G$  modulo the knowledge of the discrete spectrum.

The starting point is the determination of good neighborhoods at infinity of  $X/J$ , where  $J$  is a small compact open subgroup of  $G$ . Notice that  $G$  acts on  $X$  on the right. These neighborhoods are related to "boundary degenerations" of  $X$ . The proof of their existence is made by using wonderful compactifications

In this article we will show the existence of such neighborhoods assuming that  $\mathbf{F}$  is of characteristic different from 2 and  $X$  is symmetric. In particular, one does not assume that  $G$  is split. The main tool is the Cartan decomposition (cf. [BenO]), the definition of the constant term (cf [D]) and asymptotic properties of Eisenstein integrals due to Nathalie Lagier (cf. [L]). The use of Eisenstein integrals to prove results geometric in nature on symmetric spaces goes back to her work (cf. [L] Theorem 7). Notice that our neighborhoods at infinity are quite explicit in terms of the Cartan decomposition.

Once the existence of these neighborhoods at infinity of  $X$  is established, the analog of part 3 of [SV] is straightforward and leads to the Plancherel formula for  $X$ . Notice that our definition of normalized integrals differs slightly from the one in [SV] section 15.

Let  $\sigma$  be an involution of  $\underline{G}$  defined over  $F$ . Let  $H$  be the fixed point group of  $\sigma$  in  $G$  and let  $X = H \backslash G$ . We denote by  $X(G)$  the group of unramified characters of  $G$  and  $X(G)_\sigma$  be the connected component of 1 in  $\{\chi \in X(G) | \chi \circ \sigma = \chi^{-1}\}$ .

Let  $P$  be a  $\sigma$ -parabolic subgroup of  $G$  i.e. such that  $P$  and  $\sigma(P)$  are opposed. Let  $M := P \cap \sigma(P)$  be the  $\sigma$ -stable Levi subgroup of  $G$ . Let  $U$  ( resp.,  $U^-$  ) be the unipotent radical of  $P$  (resp.,  $P^- := \sigma(P)$ ) and let  $\delta_P$  be the modulus function of  $P$ . We define:

$$H_P = U^-(M \cap H), X_P = H_P \backslash G$$

The space  $X_P$  is called a "boundary degeneration " of  $X = H \backslash G$ . It is an important object whose role has been emphasized by Sakellaridis and Venkatesh.

Let  $P_\emptyset = M_\emptyset U_\emptyset$  be a minimal  $\sigma$ -parabolic subgroup of  $G$ . We will assume in this introduction that  $P_\emptyset H$  is the only  $(P_\emptyset, H)$ -open double coset in  $G$ . A split torus is said  $\sigma$ -split if all its elements are antiinvariant by  $\sigma$ . Let  $A_\emptyset$  be the maximal  $\sigma$ -split torus of the center of  $M_\emptyset$ . Let  $A_\emptyset^+$  be the closed positive chamber in  $A_\emptyset$  for  $P_\emptyset$ . The Cartan decomposition asserts (cf. [BenO]):

$$G = H A_\emptyset^+ \Omega,$$

for some compact subset,  $\Omega$ , of  $G$ . Let  $P$  be a  $\sigma$ -parabolic subgroup of  $G$ . If  $C > 0$ , let

$$A_\emptyset^+(P, C) := \{a \in A_\emptyset^+ | |\alpha(a)|_F > C, \alpha \text{ root of } A_\emptyset \text{ in } U\}.$$

We denote by  $\dot{1}$  (resp.,  $\dot{1}_P$ ) the image of the neutral element 1 of  $G$  in  $X$  (resp.,  $X_P$ ). The following Theorem (cf. Theorem 1) is an easy consequence of [D], Proposition 3.14.

**Theorem (Constant term map)**

There is a unique  $G$ -equivariant map  $c_P : C^\infty(X) \rightarrow C^\infty(X_P)$  with the following property. For every compact open subgroup  $J$  of  $G$ , there exists  $C > 0$  such that for all  $f \in C^\infty(X)$  which is  $J$ -invariant:

$$(c_P f)(\dot{1}_P a \omega) = f(\dot{1} a \omega), a \in A_\emptyset^+(P, C), \omega \in \Omega.$$

The following theorem (cf. Theorem 2 for its detailed version) was suggested by the work [SV] of Sakellaridis and Venkatesh, who constructed similar maps, in their context, using wonderful compactifications.

**Theorem ( $\exp_{P,J}$ -maps)**

Let  $P = MU$  be a standard  $\sigma$ -parabolic subgroup of  $G$  i.e. such that  $P_\emptyset \subset P$ . Let  $J$  be a compact open subgroup of  $G$ .

- (i) There exists  $C > 0$  such that the correspondence  $\dot{1} x J \mapsto \dot{1}_P x J$ , for  $x \in A_\emptyset^+(P, C)\Omega$ , is a well defined bijective map denoted  $\exp_{P,J}$  from the subset  $N_{X,J}(P, C) := \dot{1} A_\emptyset^+(P, C)\Omega J$  of  $X/J$ , to the subset  $O' := \dot{1}_P A_\emptyset^+(P, C)\Omega J$  of  $X_P/J$ .
- (ii) For  $J$  small enough, the map  $\exp_{P,J}$  preserves volumes.
- (iii) For  $f$  any right  $J$ -invariant element of  $C^\infty(H \backslash G)$ , one has :

$$(c_P)(\exp_{P,J}(x)) = f(x), x \in N_{X,J}(P, C).$$

As said above we need some results of N. Lagier on Eisenstein integrals that we will recall. Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . Let  $(\delta, E)$  be a unitary irreducible representation of  $M$ . Let  $\chi \in X(M)_\sigma$  and let  $\delta_\chi = \delta \otimes \chi$ . We denote by  $i_P^G \delta_\chi$  or  $\pi_\chi$  the normalized induced representation and let  $V_\chi$  denote its space.

Let  $\eta \in E^{M \cap H}$ . Let  $\chi \in X(M)_\sigma$ , sufficiently  $P$ -dominant. There is a canonical  $H$ -fixed linear form  $\xi(P, \delta_\chi, \eta)$  on  $V_\chi$ , (cf. [BD]). One defines the Eisenstein integrals on  $X$ ,  $E(P, \delta_\chi, \eta, v) \in C^\infty(X)$ ,  $v \in V_\chi$ . by:

$$E(P, \delta_\chi, \eta, v)(\dot{1} g) = \langle \xi(P, \delta_\chi, \eta), \pi_\chi(g) v \rangle, g \in G$$

Let  $A_M$  be the maximal  $\sigma$ -split torus of the center of  $M$  and let  $\mu_\delta$  be the character of  $A_M$  by which  $A_M$  acts on  $\delta$ . The following theorem is due to Nathalie Lagier. This is the analog of a Langlands lemma on asymptotics of smooth coefficients.

One says that the sequence  $(a_n)$  satisfies  $(a_n) \rightarrow_P \infty$  if  $a_n \in A_M$  and for every root  $\alpha$  of  $A_M$  in the Lie algebra of  $U$ ,  $(|\alpha(a_n)|_F)$  tends to infinity. Let us assume  $\text{Re}(\chi) \delta_P^{-1/2}$  is  $P$ -dominant. Then if  $(a_n) \rightarrow_P \infty$  the following limit exists

$$\lim_{n \rightarrow \infty} (\chi \delta_P^{-1/2})(a_n^{-1}) \mu_\delta(a_n^{-1}) E(P, \delta_\chi, \eta, v)(\dot{1} a_n) \quad (1.1)$$

and is equal to

$$< \eta, (A(P^-, P, \delta_\chi) v)(1) >,$$

where  $A(P^-, P, \delta_\chi)$  is the (converging) intertwining integral operator.

The theorem admits a variation when  $(a_n) \rightarrow_Q \infty$ , with  $P \subset Q$ . This implies easily:

**First Key Lemma**

Let us assume that  $(a_n) \rightarrow_P \infty$  and that  $(g_n)$  is a sequence in  $G$  converging to  $g$ . If  $\dot{1}a_n g_n = \dot{1}a_n$  for all  $n$ , then  $g \in U^-(M \cap H)$ .

**Second Key Lemma**

Let  $J$  be a compact open subgroup of  $G$ . Let  $(a_n) \rightarrow_P \infty, (a'_n) \rightarrow_{P'} \infty$ , for  $P, P'$   $\sigma$ -parabolic subgroups of  $G$  and let  $g, g' \in G$ . Let us assume  $\dot{1}a_n g J = \dot{1}a'_n g' J, n \in \mathbb{N}$ .

Then  $P = P'$  and a subsequence of  $(a_n^{-1}a'_n)$  is bounded.

**Definition of  $\exp_{P,J}$**  Although we gave a formula for  $\exp_{P,J}$  it is unclear that it is well defined. Let us sketch the proof that it is well defined. If it was not well defined for all  $C > 0$ , there would exist two standard  $\sigma$ -parabolic subgroups  $Q, Q'$  of  $G$  contained in  $P$ , and two sequences  $(a_n) \rightarrow_Q \infty, (a'_n) \rightarrow_{Q'} \infty, u, u' \in G$  such that

$$\dot{1}a_n u J = \dot{1}a'_n u' J$$

and

$$\dot{1}_P a_n u' J \neq \dot{1}_P a_n u' J.$$

By the Second Key Lemma, one sees that  $Q = Q'$  and, possibly extracting a subsequence, one has from the First Key Lemma:

$$\dot{1}_Q a_n u J = \dot{1}_Q a'_n u' J.$$

A trick (see below for an other occurrence of this trick) with the constant term of the characteristic function of a  $J$ -coset in  $X$  allows to show  $\dot{1}_P a_n u J = \dot{1}_P a'_n u' J$  for  $n$  large. A contradiction which proves our claim.

**Injectivity of  $\exp_{P,J}$**

One wants to prove that, for  $C$  large, if  $x, x' \in N_{X,J}(P, C)$  and  $\exp_{P,J}(x) = \exp_{P,J}(x')$ , then  $x = x'$ . One introduces the characteristic function  $f$  of  $x \subset X$  and one will use its constant term  $c_P f$ . These functions are  $J$ -invariant and their values on a  $J$ -coset makes sense. From the properties of the constant term, if  $C$  is large enough one has:

$$(c_P f)(\exp_{P,J}(x)) = f(x) = 1.$$

But, from our hypothesis one deduces :

$$(c_P f)(\exp_{P,J}(x)) = (c_P f)(\exp_{P,J}(x')).$$

Moreover by the properties of the constant term and because  $C$  is large, one has:

$$(c_P f)(\exp_{P,J}(x')) = f(x') = 1.$$

This implies that  $f(x') = 1$ , hence  $x = x'$ , as wanted.

A compact open subgroup  $J$  of  $G$  is said to have a strong  $\sigma$ -factorization for  $P_\emptyset$  if for all  $\sigma$ -parabolic subgroup  $P = MU$  which contains  $P_\emptyset$  one has:

- 1)  $J = J_U^- J_M J_U$  for all  $\sigma$ -parabolic subgroups, where  $J_M = J \cap M, \dots$ ,
- 2) For all  $a \in A_\emptyset^+, a^{-1} J_U a \subset J_U, a J_U^- a^{-1} \subset J_U^-$ .

- 3)  $J = J_H J_P$ , where  $J_H = J \cap H$ ,  $J_P = J \cap P$ .  
 4)  $J_M$  satisfies the same properties for  $P_\emptyset \cap M$ .

There are arbitrary small compact open subgroups with a strong  $\sigma$ -factorization for  $P_\emptyset$  (cf. Kato-Takano [KT1] if the residual characteristic is different from 2, [CD] in general and Lemma 6 of this article for the "strong" version).

A choice of a  $G$ -invariant measure on  $X$  determines a  $G$ -invariant measure on  $X_P$ .

**Third Key Lemma** *Let  $J$  be a compact open subgroup with a strong  $\sigma$ -factorization for  $P_\emptyset$ . Let  $a \in A_\emptyset^+$ . Then:*

$$\dot{1}aJ = \dot{1}aJ_M J_U, \dot{1}_P aJ = \dot{1}_P aJ_M J_U,$$

$$\text{vol}_X(\dot{1}aJ) = \text{vol}_{X_P}(\dot{1}_P aJ).$$

The proof is easy. Moreover one can show that the identity of volumes is also true for any small enough compact open subgroup of  $G$ . This implies easily the last property of  $\exp_{P,J}$ .

Then, one introduce the restriction  $e_P$  of the transpose map of the constant term map to  $C_c^\infty(X_P)$ . Following an idea given to us by Joseph Bernstein, and using a result of Aizebbud, Avni, Gourevitch[AAG], one shows that its image is contained in  $C_c^\infty(X)$  (cf Theorem 3). This achieves to prove the analog of Theorems 5.1.1 and 5.1.2 of [SV].

Then, as was said before, this allows to prove the Plancherel formula, modulo the discrete spectrum of the  $X_P$ , by using the same method than [SV], Part 3.

It is natural to ask about spherical varieties for general reductive groups. We think that the key point is the existence or not of a Cartan decomposition.

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## 2 Notations

If  $E$  is a vector space,  $E'$  will denote its dual. If  $T : E \rightarrow F$  is a linear map between two vector spaces,  $T^t$  will denote its transpose. If  $E$  is real,  $E_{\mathbb{C}}$  will denote its complexification. If  $G$  is a group,  $g \in G$  and  $X$  is a subset of  $G$ ,  $g.X$  will denote  $gXg^{-1}$ . If  $J$  is a subgroup of  $G$ ,  $g \in G$  and  $(\pi, V)$  is a representation of  $J$ ,  $V^J$  will denote the space of invariant elements of  $V$  under  $J$  and  $(g\pi, gV)$  will denote the representation of  $g.J$  on  $gV := V$  defined by:

$$(g\pi)(g.x) := \pi(x), x \in J.$$

We will denote by  $(\pi', V')$  the contragredient representation of a representation  $(\pi, V)$  of  $G$  in the algebraic dual vector space  $V'$  of  $V$ .

If  $V$  is a vector space of vector valued functions on  $G$  which is invariant by right (resp., left) translations, we will denote by  $\rho$  (resp.,  $\lambda$ ) the right (resp., left) regular representation of  $G$  in  $V$ .

If  $G$  is locally compact,  $d_l g$  or  $dg$  will denote a left invariant Haar measure on  $G$  and  $\delta_G$  will denote the modulus function.

Let  $\mathbf{F}$  be a non archimedean local field. We assume:

$$\text{The characteristic of } \mathbf{F} \text{ is different from } 2. \quad (2.1)$$

Let  $|\cdot|_{\mathbf{F}}$  be the normalized absolute value of  $\mathbf{F}$ .

One considers various algebraic groups defined over  $\mathbf{F}$ , and a sentence like:

$$\text{" let } A \text{ be a split torus " will mean " let } A \text{ be the group of } \mathbf{F}\text{-points of a} \quad (2.2)$$

$$\text{torus, } \underline{A}, \text{ defined and split over } \mathbf{F} \text{ " .}$$

With these conventions, let  $G$  be a connected reductive linear algebraic group. Let  $\tilde{A}_G$  be the maximal split torus of the center of  $G$ . The change with standard notation will become clear later.

Let  $\underline{G}$  be the algebraic group defined over  $\mathbf{F}$  whose group of  $\mathbf{F}$ -points is  $G$ . Let  $\sigma$  be a rational involution of  $\underline{G}$  defined over  $\mathbf{F}$ . Let  $H$  be the group of  $\mathbf{F}$ -points of an open  $\mathbf{F}$ -subgroup of the fixed point set of  $\sigma$ . We will also denote by  $\sigma$  the restriction of  $\sigma$  to  $G$ .

A split torus  $A$  of  $G$  is said  $\sigma$ -split if  $A$  is contained in the set of elements of  $G$  which are antiinvariant by  $\sigma$ . We will denote by  $A_G$  the maximal  $\sigma$ -split torus of the center of  $G$ .

If  $J$  is an algebraic subgroup of  $G$  stable by  $\sigma$ , one denotes by  $\text{Rat}(J)_{\sigma}$  the group of its rational characters defined over  $\mathbf{F}$  which are antiinvariant by  $\sigma$ . Let us define:

$$\mathfrak{a}_G = \text{Hom}_{\mathbb{Z}}(\text{Rat}(G)_{\sigma}, \mathbb{R}).$$

The restriction of rational characters from  $G$  to  $A_G$  induces an isomorphism:

$$\text{Rat}(G)_{\sigma} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \text{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \quad (2.3)$$

Notice that  $\text{Rat}(A_G)$  appears as a generating lattice in the dual space  $\mathfrak{a}'_G$  of  $\mathfrak{a}_G$  and:

$$\mathfrak{a}'_G \simeq \text{Rat}(G)_{\sigma} \otimes_{\mathbb{Z}} \mathbb{R}. \quad (2.4)$$

One has the canonical map  $H_G : G \rightarrow \mathfrak{a}_G$  which is defined by:

$$e^{\langle H_G(x), \chi \rangle} = |\chi(x)|_{\mathbf{F}}, \quad x \in G, \chi \in \text{Rat}(G)_{\sigma}. \quad (2.5)$$

The kernel of  $H_G$ , which is denoted by  $G^1$ , is the intersection of the kernels of the characters of  $G$ ,  $|\chi|_{\mathbf{F}}$ ,  $\chi \in \text{Rat}(G)_{\sigma}$ . One defines  $X(G)_{\sigma} = \text{Hom}(G/G^1, \mathbb{C}^*)$ . It is a subgroup of the group  $X(G)$  of unramified characters of  $G$ . It is precisely the connected component of the neutral element of the group of elements of  $X(G)$  which are invariant by  $\sigma$ .

One denotes by  $\tilde{\mathfrak{a}}_{G, \mathbf{F}}$  (resp.,  $\mathfrak{a}_{G, \mathbf{F}}$ ) the image of  $G$  (resp.,  $A_G$ ) by  $H_G$ . The group  $G/G^1$  is isomorphic to the lattice  $\mathfrak{a}_{G, \mathbf{F}}$ .

There is a surjective map:

$$(\mathfrak{a}'_G)_{\mathbb{C}} \rightarrow X(G)_{\sigma} \rightarrow 1 \quad (2.6)$$

denoted by  $\nu \mapsto \chi_{\nu}$  which associates to  $\chi \otimes s$ , with  $\chi \in \text{Rat}(G)_{\sigma}$ ,  $s \in \mathbb{C}$ , the character  $g \mapsto |\chi(g)|_{\mathbf{F}}^s$  (cf. [W], I.1.(1)). In other words:

$$\chi_{\nu}(g) = e^{\langle \nu, H_G(g) \rangle}, \quad g \in G, \nu \in (\mathfrak{a}'_G)_{\mathbb{C}}. \quad (2.7)$$

The kernel is a lattice and it defines a structure of a complex algebraic variety on  $X(G)_\sigma$  of dimension  $\dim_{\mathbb{R}} \mathfrak{a}_G$ . Moreover  $X(G)_\sigma$  is an abelian complex Lie group whose Lie algebra is equal to  $(\mathfrak{a}'_G)_{\mathbb{C}}$ .

If  $\chi$  is an element of  $X(G)_\sigma$ , let  $\nu$  be an element of  $\mathfrak{a}'_{G,\mathbb{C}}$  such that  $\chi_\nu = \chi$ . The real part  $\operatorname{Re} \nu \in \mathfrak{a}'_G$  is independent from the choice of  $\nu$ . We will denote it by  $\operatorname{Re} \chi$ . If  $\chi \in \operatorname{Hom}(G, \mathbb{C}^*)$  is continuous and antiinvariant by  $\sigma$ , the character of  $G$ ,  $|\chi|$ , is an element of  $X(G)_\sigma$ . One sets  $\operatorname{Re} \chi = \operatorname{Re} |\chi|$ . Similarly, if  $\chi \in \operatorname{Hom}(A_G, \mathbb{C}^*)$  is continuous, the character  $|\chi|$  of  $A_G$  extends uniquely to an element of  $X(G)_\sigma$  with values in  $\mathbb{R}^{*+}$ , that we will denote again by  $|\chi|$  and one sets  $\operatorname{Re} \chi = \operatorname{Re} |\chi|$ .

A parabolic subgroup  $P$  of  $G$  is called a  $\sigma$ -parabolic subgroup if  $P$  and  $\sigma(P)$  are opposite parabolic subgroups. Then  $M := P \cap \sigma(P)$  is the  $\sigma$ -stable Levi subgroup of  $P$ . If  $P$  is such a parabolic subgroup,  $P^-$  will denote  $\sigma(P)$ .

If  $P$  is a  $\sigma$ -parabolic subgroup of  $G$ ,  $PH$  is open in  $G$ . (2.8)

The sentence : "Let  $P = MU$  be a parabolic subgroup of  $G$ " will mean that  $U$  is the unipotent radical of  $P$  and  $M$  is a Levi subgroup of  $G$ . If moreover  $P$  is a  $\sigma$ -parabolic subgroup of  $G$ ,  $M$  will denote its  $\sigma$ -stable Levi subgroup.

If  $P = MU$  is a  $\sigma$ -parabolic subgroup of  $G$ , we keep the same notations with  $M$  instead of  $G$ .

The inclusions  $A_G \subset A_M \subset M \subset G$  determine a surjective morphism  $\mathfrak{a}_{M,\mathbf{F}} \rightarrow \mathfrak{a}_{G,\mathbf{F}}$  (resp., an injective morphism,  $\tilde{\mathfrak{a}}_{G,\mathbf{F}} \rightarrow \tilde{\mathfrak{a}}_{M,\mathbf{F}}$ ) which extends uniquely to a surjective linear map between  $\mathfrak{a}_M$  and  $\mathfrak{a}_G$  (resp., injective map, between  $\mathfrak{a}_G$  and  $\mathfrak{a}_M$ ). The second map allows to identify  $\mathfrak{a}_G$  with a subspace of  $\mathfrak{a}_M$  and the kernel of the first one,  $\mathfrak{a}_M^G$ , satisfies:

$$\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G. \quad (2.9)$$

If an unramified character of  $G$  is trivial on  $M$ , it is trivial on any maximal compact subgroup of  $G$  and on the unipotent radical of  $P$ , hence on  $G$ . This allows to identify  $X(G)_\sigma$  to a subgroup of  $X(M)_\sigma$ . Then  $X(G)_\sigma$  is the analytic subgroup of  $X(M)_\sigma$  with Lie algebra  $(\mathfrak{a}'_G)_{\mathbb{C}} \subset (\mathfrak{a}'_M)_{\mathbb{C}}$ . This follows easily from (2.7) and (2.9).

Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . Recall that  $A_M$  is the maximal  $\sigma$ -split torus of the center of  $M$ .

Let  $A_P^+$ , (resp.,  $A_P^{++}$ ) be the set of  $P$ -dominant (resp., strictly dominant) elements in  $A_M$ . More precisely, if  $\Sigma(P)$  is the set of roots of  $A_M$  in the Lie algebra of  $P$ , and  $\Delta(P)$  is the set of simple roots, one has:

$$A_P^+ \text{ (resp., } A_P^{++}) = \{a \in A_M \mid |\alpha(a)|_{\mathbf{F}} \geq 1, \text{ (resp., } > 1) \alpha \in \Delta(P)\}.$$

Let  $A_\emptyset$  be a maximal  $\sigma$ -split torus contained in  $M$ . Let  $\Sigma(U, A_\emptyset)$  be the set of roots of  $A_\emptyset$  in the Lie algebra of  $U$ , and let  $\Delta(P, A_\emptyset)$  be the set of simple roots. One defines for  $C > 0$  :

$$A_\emptyset^+(P, C) = \{a \in A_\emptyset \mid |\alpha(a)|_{\mathbf{F}} \geq C, \alpha \in \Delta(U, A_\emptyset)\}. \quad (2.10)$$

Let  $A$  be a  $\sigma$ -split torus and  $g \in G$ . We will say that  $g$  is  $A$ -good if and only if  $g^{-1} \cdot A$  is a  $\sigma$ -split torus. Let us prove:

$$\text{If } g \text{ is } A\text{-good } \sigma(g)g^{-1} \text{ commutes to } A \quad (2.11)$$

It is enough to prove that if  $a \in A$ ,  $(\sigma(g)g^{-1}).a = a$ . One has  $(\sigma(g)g^{-1}).a = \sigma(g.\sigma(g^{-1}.a)) = \sigma(g.(g^{-1}a^{-1})) = a$ .

For the rest of the article, we fix  $P_\emptyset = M_\emptyset U_\emptyset$  a minimal  $\sigma$ -parabolic subgroup of  $G$  and let  $A_\emptyset$  be the maximal  $\sigma$ -split torus of the center of  $M_\emptyset$ . It is a maximal  $\sigma$ -split torus of  $G$ . One denotes by  $A_\emptyset^+$  the set  $A_{P_\emptyset}^+$ . A  $\sigma$ -parabolic subgroup of  $G$  will be said standard (resp., semistandard) if it contains  $P_\emptyset$  (resp.,  $M_\emptyset$ ). We choose a maximal split torus  $A_0$  which contains  $A_\emptyset$ . From [HH], Lemma 1.9, it is  $\sigma$ -stable. Let  $K_0$  be the stabilizer of a special point of the apartment of the extended building of  $G$  associated to  $A_0$ .

From [BD], Lemma 2.4, there exists a finite set  $\mathcal{W}_{M_\emptyset}^G$  of  $A_\emptyset$ -good elements of  $G$ , such that if  $P$  is any semi-standard minimal  $\sigma$ -parabolic subgroup of  $G$ ,  $\mathcal{W}_{M_\emptyset}^G$  is a set of representatives of the  $(P, H)$ -double open cosets. We will assume that  $1 \in \mathcal{W}_{M_\emptyset}^G$ . (2.12)

For sake of completeness we will recall the definition of  $\mathcal{W}_{M_\emptyset}^G$ . Let  $(A_i)_{i \in I}$  be a set of representatives of the  $H$ -conjugacy classes of maximal  $\sigma$ -split torus of  $G$ . Let us assume that  $A_\emptyset$  belongs to this set. The groups  $A_i$  are conjugate under  $G$  (cf. [HH], Proposition 1.16). Let us choose for each  $i$  in  $I$ , an element  $x_i$  of  $G$ , such that  $x_i.A_\emptyset = A_i$  with  $x_\emptyset = 1$ . Let  $M_i$  be the centralizer of  $A_i$  in  $G$ . If  $L$  is a subgroup of  $G$ , one denotes by  $W_L(A_i)$  the quotient of the normalizer in  $L$  of  $A_i$  by its centralizer. Let us denote by  $W(A_i)$  instead of  $W_G(A_i)$ .

Let  $\mathcal{W}_i$  be a set of representatives in  $N_G(A_\emptyset)$  of  $W(A_\emptyset)/W_{H_i}(A_\emptyset)$  where  $H_i = x_i^{-1}.H$ . Then ([HH], Theorem 3.1) one can take  $\mathcal{W}_{M_\emptyset}^G = \cup_{i \in I} \mathcal{W}_i x_i^{-1}$ .

For  $g \in G$  we define  $\dot{g}$  the left coset  $Hg$  and we define:

$$\mathcal{X}_{M_\emptyset}^G := \{\dot{x} | x^{-1} \in \mathcal{W}_{M_\emptyset}^G\}.$$

### 3 The $G$ -spaces $X_P$ , the constant terms and the maps $c_{P,Q}$

#### 3.1 The $G$ -spaces $X_P$

One has (cf. [CD], Lemma 9.4):

Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . The union of the  $(P, H)$  open double cosets in  $G$  is equals to  $G' := \underline{P} \underline{H} \cap G$ . The set  $G'$  is also equal to the set of  $g \in G$  such that  $g^{-1}.P$  is a  $\sigma$ -parabolic subgroup. (3.1)

Let us prove:

Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$  and  $g \in G$  such that  $g.A_M$  is  $\sigma$ -split. Then  $g.P$  is a  $\sigma$ -parabolic subgroup of  $G$ . (3.2)

One has  $P = P_\nu$  for some  $\nu \in \mathfrak{a}'_M$  in the sense of [CD], (2.14). Then  $g.P_\nu = P_\mu$  where  $\mu$  is the conjugate of  $\nu$  by  $g$ . Our hypothesis implies that  $\sigma(\mu) = -\mu$ . This implies



that  $g.P_\nu$  is a  $\sigma$ -parabolic subgroup as  $P_\mu$  and  $\sigma(P_\mu) = P_{-\mu}$  are opposite parabolic subgroups.

One easily extends [CD], Equation (7.1), by replacing  $A_\emptyset$  by  $A_M$ , the proof being identical:

Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . Let  $y, y'$  be  $A_M$ -good elements of  $G$  such that  $PyH = Py'H$ . Then there exist  $m \in M, h \in H$  such that  $y' = myh$ . (3.3)

We define an equivalence relation  $\approx_M$  on  $\mathcal{X}_{M_\emptyset}^G$  by  $x \approx_M x'$  if and only if  $Px^{-1}H = Px'^{-1}H$ , which by the above equation is equivalent to  $xM = x'M$ , as  $x^{-1}, x'^{-1}$  are  $A_\emptyset$ -good. Let  $\mathcal{X}_M^G$  be a set of representatives of the equivalence classes of this relation. Let us define

$$\mathcal{W}_M^G := \{y \in \mathcal{W}_{M_\emptyset}^G \mid (y^{-1}) \in \mathcal{X}_M^G\}.$$

From the above and from (2.12) one has:

The set  $\mathcal{X}_M^G$  is a set of representatives of the open  $(H, P)$ -double cosets in  $G$ . (3.4)

**Lemma 1** *Let  $P = MU$  be a semistandard  $\sigma$ -parabolic subgroup of  $G$ .*

(i) *The set of elements  $g$  of  $G$  such that  $g.A_M$  is  $\sigma$ -split is denoted  $X_M^{Lev} \subset G$ . It is left invariant by  $H$ . Its quotient by  $H$  on the left is denoted by  $X_M \subset H \backslash G$ . One has  $\mathcal{X}_M^G \subset X_M$  and:*

$$X_M = \cup_{x \in \mathcal{X}_M^G} xM \subset H \backslash G,$$

*the union being disjoint.*

(ii) *For each  $x \in \mathcal{X}_M^G$ ,  $xM$  is closed in  $X$ .*

(iii) *We endow  $X_M$  with the topology induced by the topology of  $X$ . Then for each  $x \in \mathcal{X}_M^G$ ,  $x.M$  is open and closed in  $X_M$ . Moreover the canonical map  $(M \cap x^{-1}.H \backslash M) \rightarrow xM$ ,  $(M \cap x^{-1}.H)m \mapsto xm$ , is a homeomorphism.*

(iv) *For all  $x \in X_M$ ,  $xP$  is open in  $X$  and  $X_M P = X_M U$  is the union of the open orbits of  $P$  in  $X$ .*

*Proof :*

(i) If  $g \in X_M^{Lev}$ ,  $g.P$  is a  $\sigma$ -parabolic subgroup (cf. (3.2)). From (2.8) one has  $g^{-1} \in G'$ . One deduces from (2.12) and the definition of the relation  $\approx_M$  that  $g^{-1} \in PyH$  for some  $y \in \mathcal{W}_M^G$ . From (3.3), one deduces that there exists  $m \in M, h \in H$  such that  $g^{-1} = myh$ . The equality of (i) follows immediately. From (3.3), if  $x, x'$  are distinct elements of  $\mathcal{X}_M^G$ , the sets  $HxP$  and  $Hx'P$  are disjoint. The disjointness follows.

(ii) Changing  $H$  into  $x^{-1}.H$ , one is reduced to prove (ii) when  $x$  is equal to 1. If  $(m_n)$  is a sequence in  $M$  such that  $(m_n)$  converges in  $X$  to  $l$ , then  $(\sigma(m_n)^{-1}m_n)$  converges. The Cartan decomposition for  $M \cap H \backslash M$  (cf. [BenO] Theorem 1.1) allows to extract a subsequence of  $(m_n)$  denoted again by  $(m_n)$  such that  $m_n = h_n x a_n \omega_n$ , where  $(\omega_n)$  converges,  $a_n \in A_\emptyset$ ,  $x^{-1} \in M$  is  $A_\emptyset$ -good and  $h_n \in M \cap H$ . Then using (2.11) one has:

$$\sigma(m_n)^{-1}m_n = \sigma(\omega_n^{-1}) \sigma(x^{-1}) x a_n^2 \omega_n, n \in \mathbb{N}.$$

Hence  $(a_n^2)$  is convergent and  $(a_n)$  is bounded. Extracting again a subsequence we can assume that  $(a_n)$  is convergent. This implies that  $(M \cap H)m_n$  is convergent in

$((M \cap H) \backslash M)$  is convergent in  $M \cap H \backslash M$  and  $l$  is element of  $iM$ . This proves (ii).  
 (iii) The fact that  $xM$  is closed follows from (ii). As  $\mathcal{X}G_M$  is finite, (i) implies that  $xM$  is open in  $X_M$ . The last assertion follows from [BD], Lemma 3.1 (iii).  
 (iv) From (3.1) and (3.2) and the definition of  $X_M$ , one sees that  $HxP$  is open in  $G$ . This achieves to prove the first assertion of (iv). The second follows from this and from (3.4).  $\square$

**Definition 1** *Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . Then  $X_M$  is a  $P^-$ -space with the given action of  $M$  and with the trivial action of  $U^-$ . We define:*

$$X_P = X_M \times_{P^-} G.$$

*Then  $X_M$  identifies to a subset of  $X_P$ . If  $x \in X_M$ , its image in  $X_P$  will be denoted by  $x_P$ .*

*If  $x, x' \in X_M$  the notation  $x \approx_M x'$  will mean that  $x, x'$  are in the same  $M$ -orbit in  $X_M$ . The following assertion follows from the definition of  $X_P$ .*

Let  $x, x' \in X_M$ . The following conditions are equivalent:  
 (i)  $x_P G = x'_P G$ .  
 (ii)  $xM = x'M$  in other words  $x \approx_M x'$ . (3.5)

We define  $H_P := U^-(M \cap H)$ . If  $y \in G$ , let us denote by  $\sigma_y$  the rational involution of  $G$  defined by:

$$\sigma_y(g) = y^{-1} \sigma(ygy^{-1})y,$$

whose fixed point set is equal to  $y^{-1}.H$ . Moreover  $\sigma_y$  depends only on  $y$ .

Let  $x \in X_M \subset H \backslash G$ . The stabilizer of  $x_P$  in  $G$  is equal to  $(x^{-1}.H)_P := U^-(M \cap x^{-1}.H)$ . (3.6)

**Definition 2** *Let  $a \in A_M$ . From (3.5) any element  $y \in X_P$  is of the form  $y = x_P g$  for a unique element  $x \in \mathcal{X}_M^G$  and some element  $g \in G$ , which is defined up to the left action of  $U^-(M \cap x^{-1}.H)$ . We see easily from (3.6) that  $a.y := x_P a g$  is well defined. It defines a left action of  $A_M$  on  $X_P$  which commutes to the right  $G$ -action.*

From the equality in Lemma 1, one deduces the following equality:

**Lemma 2** (i) *One has:*

$$X_P = \cup_{x \in \mathcal{X}_M^G} x_P G, \tag{3.7}$$

*the union being disjoint.*

(ii) *For  $x \in \mathcal{X}_M^G$ ,  $x_P P$  is the unique open orbit in  $x_P G$ .*

(iii) *Let  $(X_M)_P$  the image of  $X_M$  in  $X_P$  or equivalently the set  $\{x_P | x \in X_M\}$ . The union of the open  $P$ -orbits in  $X_P$  is equal to  $(X_M)_P P = (X_M)_P U$  and the map from  $X_M P = X_M U$  to  $(X_M)_P P$  defined by  $xu \mapsto x_P u$  is a bijective  $P$ -equivariant map.*

*Proof :*

One deduces (i) from the equality in Lemma 1.

(ii) It follows from (3.6) that  $x_P G$  is isomorphic to  $U^-(M \cap x^{-1}.H) \backslash G$ . Then (ii) follows from the fact that there is a unique open  $(U^-, P)$ -double coset in  $G$ .

The first part of (iii) is clear. It follows from (3.6) that the map  $X_M \times U \rightarrow (X_M)_P P$ ,  $(x, u) \mapsto (x_P u)$  is bijective. One checks easily that it is  $P$ -equivariant.  $\square$

Let  $P$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . Let us prove:

$$\{a \in A_\emptyset \mid |\alpha(a)|_{\mathbf{F}} \geq C, \alpha \in \Delta(P_\emptyset \cap M)\} = A_\emptyset^+(P_\emptyset, C) A_M. \quad (3.8)$$

The right hand side is clearly included in the left hand side of the equality to prove. Let  $a$  be an element of the left hand side. Let  $b \in A_M$  be strictly  $P$ -dominant. Then for large  $n \in \mathbb{N}$ , one has  $ab^n \in A_\emptyset^+(P_\emptyset, C)$ . Our claim follows.

**Proposition 1** *There exists a compact subset  $\Omega$  of  $G$  such that for all  $\sigma$ -parabolic subgroup  $P$  of  $G$  containing  $M_\emptyset$ , one has:*

$$X_P = \cup_{x \in \mathcal{X}_{M_\emptyset}^G} x_P A_\emptyset^+ A_M \Omega.$$

*Proof :*

The claim is true for  $P = G$  from the Cartan decomposition for symmetric space (cf. [BenO] Theorem 1.1). In general one has  $G = P^- K_0$  hence

$$X_P = X_M P^- K_0 = X_M K_0 = \cup_{x \in \mathcal{X}_M^G} x_P M K_0.$$

The  $M$ -space  $xM \subset H \backslash G$  is a symmetric space for  $M$  for the involution  $\sigma_x$  restricted to  $M$ . As  $x$  is  $A_\emptyset$  good,  $P_\emptyset \cap M$  is a  $\sigma_x$ -parabolic subgroup of  $M$  (cf. [CD] Lemma 2.2). From the Cartan decomposition for this symmetric space, it is enough to prove the following lemma.

**Lemma 3** *The open orbits of  $P_\emptyset \cap M$  in  $xM$  are the orbits  $y(P_\emptyset \cap M)$ , where  $y$  describes the set of elements in  $\mathcal{X}_{M_\emptyset}^G$  such that  $y \approx_M x$ .*

By conjugating on the left by  $x^{-1}$  and changing  $H$  into  $x^{-1}.H$  one is reduced to prove the lemma for  $x = \dot{1}$ . Any open  $(P_\emptyset \cap M)$ -orbit in  $(M \cap H) \backslash M$  is of the form  $(M \cap H)z(P_\emptyset \cap M)$  where  $z^{-1}$  is  $A_\emptyset$ -good and element of  $M$  (cf. (2.12)). As  $HP$  is open, the product map  $H \times P \rightarrow HP$  is open (cf. [BD], Lemma 3.1 (iii)). Hence, as  $H z P_\emptyset = H((H \cap M)z(P_\emptyset \cap M))U$ , one sees that  $H z P_\emptyset$  is open. Then (2.12) implies the existence of an element  $y$  of  $\mathcal{X}_{M_\emptyset}^G$  such that:

$$H z P_\emptyset = H y P_\emptyset. \quad (3.9)$$

As  $z \in M$ ,  $z$  is  $A_M$ -good. As  $y$  is also  $A_M$ -good, it follows from (3.3) that  $z = h y m'$  for some  $m' \in M$ ,  $h \in H$  and one has  $y \approx_M z$ . Let us prove:

$$(H z P_\emptyset) \cap H M = H z (P_\emptyset \cap M).$$

Let  $p \in P_\emptyset$  and let us write  $p = p'u$  with  $p' \in P_\emptyset \cap M$  and  $u \in U$ . Let us show that  $zp'u \in HM$  if and only if  $u = 1$ . Let  $m' := zp' \in M$ . If  $zp'u \in HM$ , there exist  $h \in H$ ,  $m \in M$  such that  $m'u = hm$ . Then one has

$$h = m'm^{-1}(m.u)$$

Hence both sides of the equality are elements of  $H \cap P = H \cap M$ . It follows that  $u = 1$ . Our claim follows.

As  $z \in M$ ,  $z \approx_M \dot{1}$ . Taking into account  $y \approx_M z$ , one has  $y \approx_M \dot{1}$  and one shows similarly that:

$$(HyP_\emptyset) \cap HM = Hy(P_\emptyset \cap M). \quad (3.10)$$

From this and (3.9) one sees that:

$$Hy(P_\emptyset \cap M) = Hz(P_\emptyset \cap M).$$

This shows that any open  $P_\emptyset \cap M$ -orbit in  $\dot{1}M$  has the required form.

Reciprocally from (3.10) one sees that for all  $y \in \mathcal{X}_{M_\emptyset}^G$  such that  $y \approx_M \dot{1}$ ,  $y(P_\emptyset \cap M)$  is open in  $X_M$  as it is equal to the intersection of an open set of  $X$  with the open subset  $\dot{1}M$  of  $X_M$  (cf. Lemma 1 (iii)). This proves the Lemma.  $\square$

**Remark 1** 1) *There is a minor change with [SV]. Here we are interested to  $X = H \backslash G$  but Sakellaridis and Venkatesh study the bigger space  $(\underline{H} \backslash \underline{G})(\mathbf{F})$ . The space  $X$  appears as one of the finitely many  $G$ -orbits in  $\underline{X}(\mathbf{F})$  and every  $G$ -orbit in  $\underline{X}(\mathbf{F})$  is of the same type than  $X$ .*

2) *If  $P = MU$  is a standard  $\sigma$ -parabolic subgroup of  $G$ , we define  $\Theta_P$  as the set of simple  $A_\emptyset$ -roots in the Lie algebra of  $M$  which are simple for  $P_\emptyset$ . Notice that  $\Theta_{P_\emptyset} = \emptyset$ . We could define also  $A_{\Theta_P} = A_P$ . Then  $A_{\Theta_P}$  plays here the role  $A_{X, \Theta_P}$  in [SV].*

### 3.2 Constant term

Let  $J$  be a totally discontinuous group acting continuously on a totally disconnected topological space  $Y$ . We will say that the action is smooth if the stabilizer of any element of  $Y$  is open and we will denote by  $C^\infty(Y)$  the space of functions which are fixed by the right action of some compact open subgroup of  $G$ .

Let us recall (cf. [D], Proposition 3.14) the following result.

Let  $P = MU$  be a  $\sigma$ -parabolic subgroup of  $G$ . Let  $(\pi, V)$  be a smooth  $G$ -submodule of  $C^\infty(H \backslash G)$ . The map  $f \rightarrow f_P$  is the unique morphism of  $P$ -modules from  $V$  to the space  $C^\infty((M \cap H) \backslash M)$  endowed with the right action of  $M$  tensored by  $\delta_P^{1/2}$  and the trivial action of  $U$ , such that:

For all compact open subgroup,  $J$ , of  $G$  there exists  $C > 0$ , such that for all  $f \in V^J$ :

$$f(a) = \delta_P^{1/2}(a)f_P(a), a \in A_M(P^-, C) = A_M \cap A_\emptyset^+(P^-, C), \quad (3.11)$$

where  $A_\emptyset^+(P^-, C)$  has been defined in (2.10).

We have a similar statement by replacing the preceding equality by

$$f(a) = \delta_P^{1/2}(a)f_P(a), a \in A_\emptyset^+(P^-, C)$$

We have slightly modified the statement of l.c. by replacing  $A_0$  by  $A_M$  and  $A_\emptyset$  but unicity still holds due to [D] Equation (3.8). It is useful to introduce:

$$\tilde{f}_P = \delta_P^{1/2} f_P. \quad (3.12)$$

Let us assume that  $V$  is of finite length. Let  $(\delta, E)$  be the unnormalized Jacquet module of  $V$ . Then there exists a finite family of complex characters  $\chi_1, \dots, \chi_r$  of  $A_M$  such that

$$(\delta(a) - \chi_1(a)) \dots (\delta(a) - \chi_r(a)) = 0, a \in A_M \quad (3.13)$$

From the intertwining properties of the map  $f \mapsto f_P$ , one deduces

$$(\rho(a) - \chi_1(a)) \dots (\rho(a) - \chi_r(a)) \tilde{f}_P = 0, a \in A_M.$$

**Theorem 1** *Let  $P = MU \subset Q = LV$  be two standard  $\sigma$ -parabolic subgroups of  $G$ . If  $C \geq 0$ , let  $A_\emptyset^+(P, Q, C)$  be the set of  $a \in A_\emptyset^+$  such that  $|\alpha(a)|_{\mathbf{F}} \geq C$  for all roots  $\alpha$  of  $A_\emptyset$  in the Lie algebra of  $U \cap L$ .*

*(i) There exists a unique  $G$ -equivariant map  $c_{P,Q}$  from  $C^\infty(X_Q)$  to  $C^\infty(X_P)$  satisfying the following property:*

*For all compact open subgroups  $J$  of  $G$ , there exists  $C > 0$  such that for all  $f \in C^\infty(X_Q)$  which is right  $J$ -invariant, one has:*

$$(c_{P,Q}f)(x_P a) = f(x_Q a), a \in A_\emptyset^+(Q, P, C), x \in \mathcal{X}_{M_\emptyset}^G. \quad (3.14)$$

*The map does not depend on the choice of  $\mathcal{W}_{M_\emptyset}^G$ .*

*(ii) Let  $R$  be an other standard  $\sigma$ -parabolic subgroup of  $G$  such that  $Q \subset R$ . Then one has:*

$$c_{P,R} = c_{P,Q} \circ c_{Q,R}.$$

*(iii) Let  $\mathcal{V}$  be a smooth  $G$ -submodule of finite length of  $C^\infty(X_Q)$ . Then there exists a finite family of complex characters  $\chi_1, \dots, \chi_r$  such that for all  $f \in \mathcal{V}$ :*

$$((\lambda(a) - \chi_1(a)) \dots (\lambda(a) - \chi_r(a)) c_{P,Q}f)(x_P g) = 0, x \in \mathcal{X}_M^G, g \in G, a \in A_M.$$

For the proof we will need two lemmas.

**Lemma 4** *Let  $x \in X_M$ .*

*(i) If  $f \in C^\infty(X_Q)$  and  $g \in G$ , let  $f_{x_Q, g}$  be the map  $l \mapsto f(x_Q l g)$  viewed as a map on  $(x^{-1}.H) \cap L \setminus L$ . We define a function  $f_{x_Q, P \cap L}$  on  $G$  by  $g \mapsto (f_{x_Q, g})_{P \cap L}^\sim(1)$ , where we use the notation (3.12). It is left invariant by  $(x^{-1}.H)_P$  and it is right  $J$ -invariant if  $f$  is right  $J$ -invariant.*

*(ii) The point (i) allows to define a map  $c_{P,Q,x} : C^\infty(X_Q) \rightarrow C^\infty(x_P G)$  by*

$$(c_{P,Q,x}f)(x_P g) = f_{x_Q, P \cap L}(g).$$

*It intertwines the right regular representations of  $G$  on  $C^\infty(X_Q)$  and  $C^\infty(x_P G)$ .*

*(iii) One has*

$$(c_{P,Q,x}f)(x_P m g) := (f_{x_Q, g})_{P \cap L}^\sim(m), m \in M.$$

(iv) For all compact open subgroup  $J$  of  $G$ , there exists  $C > 0$  such that for all  $x \in \mathcal{X}_{M_0}^G$ , for all  $f \in C^\infty(X_Q)$  which is right  $J$ -invariant, one has:

$$(c_{P,Q,x}f)(x_Pa) = f(x_Qa), a \in A_\emptyset^+(P, Q, C), x \in \mathcal{X}_{M_0}^G.$$

(v) We have unicity of the  $G$ -maps satisfying the condition above on the sets  $A_M \cap A_\emptyset^+(P, Q, C)$ .

*Proof :*

(i) Due to the intertwining properties of the constant term map (cf. (3.11)) the map  $\varphi \mapsto \tilde{\varphi}_{P \cap L}$  intertwines the right regular representations of  $P^- \cap L$  on  $C^\infty((x^{-1}.H) \cap L) \setminus L$  and on  $C^\infty((x^{-1}.H) \cap M) \setminus M$ , where  $U^- \cap L$  acts trivially on the latter space. Also one remarks that  $f_{x_Q,vg} = f_{x_Q,g}$  for  $g \in G, v \in V$ . Altogether this shows (i) and that the map  $c_{P,Q,x}$  is well defined. The map  $c_{P,Q,x}$  intertwines the right regular representations of  $G$  as the equality  $(c_{P,Q,x}(\rho(g)f))(x_Pg') = (c_{P,Q,x}f)(x_Pg'g)$  follows from the definitions. This achieves to prove (ii).

(iii) By (ii), it is enough to prove this for  $g = 1$ . The intertwining properties of the map  $\varphi \mapsto \tilde{\varphi}_{P \cap L}$  described above allows to prove (iii).

(iv) By (iii) and from the second equality of (3.11) for  $P^- \cap L$ , one deduces (iv).

(v) As  $c_{P,Q,x}$  is a  $G$ -map, (v) follows from the second characterization in (3.11) of the constant term.  $\square$

**Lemma 5** *Let  $x, y \in X_M$ . If  $x \approx_M y$ , one has  $c_{P,Q,x} = c_{P,Q,y}$ .*

From Lemma 4 (v), it is enough to prove the following assertion.

Let  $J$  be a compact open subgroup of  $G$ . There exists  $C > 0$  such that for all  $f \in C^\infty(X_Q)$  which is  $J$ -invariant

$$(c_{P,Q,x}f)(y_Pa) = f(y_Qa), a \in A_M \cap A_\emptyset^+(P, Q, C). \quad (3.15)$$

Let  $m \in M$  such that  $y = xm$ . Then  $y_P = x_Pm, y_Q = x_Qm$ . By the interwinning properties of  $c_{P,Q,x}$  and the commutation of  $a \in A_M$  with  $m$ , one has

$$(c_{P,Q,x}f)(y_Pa) = c_{P,Q,x}(\rho(m)f)(x_Pa), a \in A_M. \quad (3.16)$$

One remarks that  $\rho(m)f$  is fixed by  $m.J$ . Hence as  $x$  satisfies Lemma 4 (iv), there exists  $C > 0$  such that for all  $f \in C^\infty(X_Q)$  right invariant by  $J$ :

$$c_{P,Q,x}(\rho(m)f)(x_Pa) = (\rho(m)f)(x_Qa), a \in A_M \cap A_\emptyset^+(P, Q, C).$$

As  $(\rho(m)f)(x_Qa) = f(y_Qa)$ , together with (3.16) this proves (3.15) and the lemma.  $\square$

*Proof of Theorem 1*

(i) We define  $c_{P,Q}(f)$  for  $f \in C^\infty(X_Q)$  by:

$$(c_{P,Q}f)(x_Qg) := (c_{P,Q,x}f)(x_Qg), x \in \mathcal{X}_M^G$$

From Lemma 4 (iv) and (v), one sees that this is well defined and that it has the required properties including unicity. Also from Lemma 5,  $c_{P,Q}$  does not depend on the choice of  $\mathcal{X}_M^G$  in  $\mathcal{X}_{M_0}^G$ . Also, as changing our choice of  $\mathcal{X}_{M_0}^G$  involves only right multiplication by elements of  $M_\emptyset$ , one sees that  $c_{P,Q}$  even does not depend of the choice of  $\mathcal{X}_{M_0}^G$ .

(ii) follows easily from the unicity statement in (i).

(iii) We use the notation of Lemma 4 (i). The map  $f \mapsto f_{x_Q,1}$  is a  $Q^-$ -map from  $\mathcal{V}$  to a  $Q^-$ -submodule of  $C^\infty((x^{-1}.H) \cap L \backslash L)$  endowed with the right action of  $L$  and the trivial action of  $V$ . This submodule is a quotient of the unnormalized Jacquet module of  $\mathcal{V}$  for  $Q^-$ . Hence it is an  $L$ -module of finite length. Then (iii) follows from the definition of  $c_{P,Q}$  above and from (3.13) applied to  $L$  instead of  $G$ .  $\square$

## 4 Neighborhoods at infinity of $X_Q$ and mappings

$exp_{X_P, X_Q}$

### 4.1 Choice of measures

We fix on  $G$  (resp.,  $H$ , resp., the unipotent radical of a semistandard  $\sigma$ -parabolic  $P = MU$  of  $G$ ) the Haar measure such that its intersections with  $K_0$  is of volume 1. From this we deduce a measure on  $H \backslash G$ . We choose the Haar measure on  $M$  such that:

$$\int_G f(g)dg = \int_{U \times M \times U^-} f(umu^-) \delta_P(m)^{-1} dudmdu^-, f \in C_c^\infty(G). \quad (4.1)$$

Also there exists a constant  $\gamma(P)$  such that:

$$\int_G f(g)dg = \gamma(P) \int_{U^- \times M \times K_0} f(u^-mk) du^- dm dk. \quad (4.2)$$

The set  $X_M U$  is an open subset of  $H \backslash G$  (cf. Lemma 1 (iv)) which is right invariant by  $P$ . Hence the measure on  $H \backslash G$  induces a right  $P$ -invariant measure on  $X_M U$ . But the map  $X_M \times U \rightarrow X_M U$ ,  $(x, u) \mapsto xu$  is a homeomorphism. As the Haar measure on  $U$  has been fixed, there is a canonical measure  $m_{X_M}$  on  $X_M$  such that:

$$\int_{X_M U} f(y)dy = \int_{X_M \times U} f(xu) dm_{X_M}(x) du, f \in C_c(X_P). \quad (4.3)$$

One checks easily that this measure satisfies:

$$\int_{X_M} f(xm) dm_{X_M}(x) = \delta_P(m)^{-1} \int_{X_M} f(x) dm_{X_M}(x), m \in M \quad (4.4)$$

Let  $x \in X_M$ . As  $U^- P$  is open in  $G$ ,  $x_P P$  is an open set in  $X_P$  which depends only on  $xM$ . By looking to the stabilizer of  $x$  and  $x_P$  one sees that the map  $x_P \mapsto x_P P$  is a well defined continuous bijection between  $xP$  and  $x_P P$  which depends only on  $xM$  hence on  $x_P P$ . Thus, our choice of  $P$ -invariant measure on  $xP$  induces and "by transport de structure" a  $P$ -invariant measure on  $x_P P$ . We fix on  $x_P G$  the  $G$ -invariant measure

which agrees with this measure on  $x_P P$ . Hence we have a right invariant measure by  $G$  on  $X_P$ . We want to deduce from  $m_{X_M}$  an  $M$ -invariant measure on  $X_M$ . This will depend on our choice of  $\mathcal{X}_M^G$ . If  $x \in \mathcal{X}_M^G$ , the map  $(M \cap x^{-1}.H) \backslash M \rightarrow xM$ ,  $(M \cap x^{-1}.H)m \mapsto xm$  is a homeomorphism (cf. e.g. [BD] Lemma 3.1 (iii)). The measure on  $X_M$  determines a measure on  $(M \cap x^{-1}.H) \backslash M$ . Let us show:

$$\text{The function } \delta_P \text{ is trivial on } M \cap x^{-1}.H. \quad (4.5)$$

The group  $P$  is a  $\sigma_x$ -parabolic subgroup of  $G$  (cf. [CD], Lemma 2.2 (iii) where one has to change  $x$  in  $x^{-1}$ ). This implies that  $\delta_P$  is antiinvariant by  $\sigma_x$  and hence trivial on the fixed points of  $\sigma_x$ . This proves our claim. This determines ‘par transport de structure’ a function denoted  $\delta_{P,x}$  on  $xM$ . Multiplying the restriction to  $xM$  of the canonical quasiinvariant measure  $m_{X_M}$  by  $\delta_{P,x}$  one gets an  $M$ -invariant measure on  $xM$  and on  $(M \cap x^{-1}.H) \backslash M$ . Hence one has:

$$\text{Our choice of } \mathcal{X}_M^G \text{ determines an } M\text{-invariant measure on } X_M. \quad (4.6)$$

It allows to identify  $C^\infty(X_M)$  to a subspace of the dual of  $C_c^\infty(X_M)$  (we will see later that this subspace of the dual is the full smooth dual, cf. after (8.1)).

One deduces also a measure on  $x^{-1}.H$  by conjugacy. Together with our choice of measure on  $M$  and on  $(M \cap x^{-1}.H) \backslash M$ , this determines a measure on  $(M \cap x^{-1}.H) \backslash x^{-1}.H$ .

We introduce a unitary action  $\mathcal{L}$  of  $A_M$  (cf. (4.4) for unitarity) on the space  $L^2(X_P)$  called normalized action:

$$\mathcal{L}_a f(x) = \delta_P^{1/2}(a) f(ax), x \in X_P, \quad (4.7)$$

where  $ax$  is the left action of  $a \in A_M$  on  $x \in X_P$  of Definition 2.

## 4.2 Compact open subgroups with a $\sigma$ -factorization

First we give a definition.

A compact open subgroup  $J$  of  $G$  is said to have a  $\sigma$ -factorization (resp. strong  $\sigma$ -factorization) for standard  $\sigma$ -parabolic subgroups of  $G$  if it satisfies the following conditions:

- (i) For every standard  $\sigma$ -parabolic subgroup  $P = MU$  of  $G$  the product map  $J_{U^-} \times J_M \times J_U \rightarrow J$  is bijective, where  $J_{U^-} = J \cap U^-$ ,  $J_M = J \cap M$ ,  $J_U = J \cap U$ .
- (ii) Let  $A \subset A_\emptyset$  be the maximal  $\sigma$ -split torus of the center of  $M$  and let  $A^-$  (resp.  $A_\emptyset^-$ ) be the set of its  $P$ -(resp.  $P_\emptyset$ )-antidominant elements. For all  $a$  belonging to  $A^-$  (resp.  $A_\emptyset^-$  for the strong  $\sigma$ -factorization) one has

$$aJ_U a^{-1} \subset J_U, a^{-1}J_{U^-} a \subset J_{U^-}.$$

- (iii) One has  $J = J_H J_P$ , where  $J_H = J \cap H$ ,  $J_P = J \cap P$ .
- (iv) For every  $\sigma$ -parabolic subgroup  $P = MU$  of  $G$  which contains  $P_\emptyset$ ,  $J \cap M$  enjoys the same properties that  $J$  for  $M$  and  $P_\emptyset \cap M$ .



From [CD] Prop 2.3, there exist arbitrary small compact open subgroups of  $G$  with a  $\sigma$ -factorization. We will need the following lemma later.

**Lemma 6** *There exists a basis of neighborhood of the identity in  $G$ ,  $(J'_n)_{n \in \mathbb{N}}$ , made of a decreasing sequence of compact open subgroups of  $G$  with a strong  $\sigma$ -factorization and such that for all  $n \in \mathbb{N}$ ,  $J'_n$  is a normal subgroup of  $J'_0$ .*

*Proof :*

We keep the notation of [CD] Prop 2.3, Then, as the basis of  $\underline{u}_\emptyset$  and  $\underline{u}_\emptyset^-$  is made of weight vectors  $\underline{a}_\emptyset$ , one has:

$$\Lambda \underline{g} = \Lambda \underline{u} \oplus \Lambda \underline{m} \oplus \Lambda \underline{u}^-,$$

where  $\Lambda \underline{u} = \Lambda \underline{g} \cap \underline{u}$ ,  $\Lambda \underline{m} = \Lambda \underline{g} \cap \underline{m}$ ,  $\Lambda \underline{u}^- = \Lambda \underline{g} \cap \underline{u}^-$  and  $\Lambda \underline{u}$  (resp.,  $\Lambda \underline{u}^-$ ) is stable by the adjoint action of  $A_\emptyset^-$  (resp.,  $A_\emptyset^+$ ). Then one shows as in the proof of [CD] Proposition 2.3, where only (ii) has to be modified, that there exists a basis of neighborhoods  $(J_n)_{n \in \mathbb{N}}$  of the identity in  $G$  made of a decreasing sequence of compact open subgroups of  $G$  with a strong  $\sigma$ -factorization.

As  $\Lambda \underline{g}$  is compact and open in  $\underline{g}$ , there exists  $n_0 \in \mathbb{N}$  such that the adjoint action of  $J_{n_0}$  preserves  $\Lambda \underline{g}$ . Hence by l.c. Lemma 10.1 (iii), there exists  $N \in \mathbb{N}$  such that for all  $n$  greater than  $\bar{N}$ ,  $J_{n_0}$  normalizes  $J_n$ . The sequence  $(J'_n)$  defined by  $J'_n = J_{N+n}$  has the required properties.  $\square$

### 4.3 Statement of Theorem 2

**Theorem 2** *Let  $P = MU \subset Q = LV$  two standard  $\sigma$ -parabolic subgroups of  $G$ . Let  $K$  be a compact open subgroup of  $G$  having a strong  $\sigma$ -factorization. Let  $\Omega$  be as in Proposition 1. We may and will assume that  $K \subset \Omega$  and that  $\Omega$  is biinvariant by  $K$ . Let  $J$  be a compact open subgroup of  $G$  such that for all  $\omega$  in  $\Omega$ ,  $x^{-1} \in \mathcal{W}_{M_\emptyset}^G$ ,  $(x\omega).J \subset K$ .*

*We define for  $C > 0$  and  $x \in \mathcal{X}_M^G$ :*

$$N_{X_Q}(x, P, C) := \cup_{y \in \mathcal{X}_{M_\emptyset}^G, y \approx_M x} y_Q A_\emptyset^+(P, Q, C) \Omega.$$

*Then there exists  $C > 0$  such that:*

*(i) The union*

$$N_{X_Q}(P, C) := \cup_{x \in \mathcal{X}_M^G} N_{X_Q}(x, P, C)$$

*is disjoint.*

*(ii) The subset  $N_{X_Q}(P, C)$  of  $X_Q$  is right  $J$ -invariant. We view  $N_{X_Q, J}(P, C) := N_{X_Q}(P, C)/J$  as a subset of  $X_Q/J$ . The map  $N_{X_Q, J}(P, C) \rightarrow X_P/J$  which associates  $x_P a \omega J$  to  $x_Q a \omega J$  with  $x \in \mathcal{X}_{M_\emptyset}^G$ ,  $a \in A_\emptyset^+(P, Q, C)$ ,  $\omega \in \Omega$  is well defined on  $N_{X_Q, J}(P, C)$ . It is denoted  $\exp_{X_P, X_Q, J}$ . The image by  $\exp_{X_P, X_Q, J}$  of  $N_{X_Q, J}(P, C)$  is equal to  $N_{X_P, J}(P, C)$*

*(iii) The map  $\exp_{X_P, X_Q, J}$  is injective on  $N_{X_Q, J}(P, C)$ .*

*(iv) As a map from a set of  $J$ -invariant subsets of  $X_Q$  to a set of  $J$ -invariant subsets of  $X_P$ ,  $\exp_{X_P, X_Q, J}$  preserves volumes.*

From the definitions, one sees:

**Corollary 1** *If  $a \in A_L$  is  $Q$ -dominant and  $z \in N_{Q,J}(P, C)$ , one sees from the definitions that  $az \in N_{X_Q,J}(P, C)$  and that:*

$$\exp_{X_P, X_Q, J}(az) = a \exp_{X_P, X, J}(z), z \in N_{X_Q, J}(P, C)$$

*First reduction for the proof of Theorem 2*

We will reduce the proof of the theorem to the case where  $Q = G$ . The proof when  $Q = G$  will be done in section 6. Let us assume that the theorem has been proved for  $Q = G$ . Let us prove it for arbitrary  $Q$ .

We will define  $\exp_{X_P, X_Q, J}$  and prove part (ii) of Theorem 2. We define  $N'_{X_Q, J}(P, C) = \exp_{X_Q, X, J}(N_{X, J}(P, C))$  which is well defined for  $C$  large. Then, from (3.8), the definition of the left  $A_L$ -action (cf. Definition 2) and the definition of  $\exp_{X_Q, X, J}$  one has:

$$N_{X_Q, J}(P, C) = A_L N'_{X, J}(P, C).$$

Let  $y \in N_{X_Q, J}(P, C)$ . By the above equality and the definition of  $N'_{X_Q, J}(P, C)$ , there exist  $a \in A_L$  and  $z \in N_{X, J}(P, C)$  such that  $y = a \exp_{X_Q, X, J}(z)$ . Let us prove that  $a \exp_{X_P, X, J}(z)$  does not depend on the choice of  $a$  and  $z$  as above.

Let us assume that there exists  $a' \in A_L$  and  $z' \in N_{X, J}(P, C)$  with  $y = a' \exp_{X_Q, X, J}(z')$ . By choosing  $b \in A_L$  sufficiently  $Q$ -dominant we can assume that  $ba, ba'$  are  $Q$ -dominant. As  $z \in N_{X, J}(P, C)$  one may write  $z = xa_z \omega J$  for some  $x \in \mathcal{X}_{M_\emptyset}^G$ ,  $a_z \in A_\emptyset^+(P, C)$ ,  $\omega \in \Omega$ . By abuse of notation, as it may depends on this writing, one defines  $baz := xbaa_z \omega J$ . One defines similarly  $b'a'z'$ . Then  $baz, b'a'z' \in N_{X, J}(P, C)$ . From our hypothesis one has:

$$ba \exp_{X_Q, X, J}(x) = ba' \exp_{X_Q, X, J}(x').$$

From Corollary 1 of Theorem 2 for  $Q = G$ , one has:

$$ba \exp_{X_Q, X, J}(z) = \exp_{X_Q, X, J}(baz), ba' \exp_{X_Q, X, J}(z) = \exp_{X_Q, X, J}(ba'z').$$

From the injectivity in (iii) for  $Q = G$ , one deduces:

$$baz = ba'z'.$$

One sees from the definition of  $\exp_{X_P, X, J}$  in (ii) that:

$$\exp_{X_P, X, J}(baz) = ba \exp_{X_P, X, J}(z), \exp_{X_P, X, J}(ba'z') = ba' \exp_{X_P, X, J}(z').$$

As  $baz = ba'z'$ , one deduces from this the equality:

$$a \exp_{X_P, X, J}(z) = a' \exp_{X_P, X, J}(z').$$

This proves our claim and it allows to define

$$\exp_{X_P, X_Q, J}(y) := a \exp_{X_P, X, J}(z).$$

Let  $y = x_Q a \omega J \in N_{X_Q, J}(P, C)$  with  $x \in \mathcal{X}_{M_\emptyset}^G$ ,  $a \in A_\emptyset^+(P, Q, C)$ . By choosing  $b' \in A_L$  sufficiently  $Q$ -dominant, one has  $a' := b'a \in A_\emptyset^+(P, C)$ . Let  $b = b'^{-1}$  and  $y' = x_Q a' \omega J$ . One has  $y = by'$  and  $y' = \exp_{X_Q, X, J}(xa' \omega J)$ . Our definition of  $\exp_{X_P, X_Q, J}$  shows that:

$$\exp_{X_P, X_Q, J}(x_Q a \omega J) = b x_P a' \omega J = x_P a \omega J.$$

This achieves to prove that  $\exp_{X_P, X_Q, J}$  is defined by the formula given in the theorem. This implies that the the image of  $N_{X_Q, J}(P, C)$  is clearly  $N_{X_P, J}(P, C)$ . This achieves the proof of Theorem 2 (ii) and Corollary 1 follows.

(iii) Let  $y, y' \in N_{X_Q, J}(P, C)$  with  $\exp_{X_P, X_Q, J}(y) = \exp_{X_P, X_Q, J}(y')$ . One wants to prove that  $y = y'$ . By multiplying  $y$  and  $y'$  by a sufficiently  $Q$ -dominant element of  $A_L$ , one may assume that  $y, y' \in N'_{X_Q, J}(P, C)$ . Then  $y = \exp_{X_Q, X, J}(z), y' = \exp_{X_Q, X, J}(z')$  with  $z, z' \in N_{X, J}(P, C)$ . From our definition of  $\exp_{X_P, X_Q, J}$ , one deduces the equality:

$$\exp_{X_P, X, J}(z) = \exp_{X_P, X, J}(z').$$

From the injectivity of  $\exp_{X_P, X, J}$  one sees that  $z = z'$ , hence  $y = y'$ . This achieves to prove (iii).

(iv) One has the equality

$$\text{vol}_{X_P}(axJ) = \delta_P(a)\text{vol}_{X_P}(xJ), x \in X_P, a \in A_P.$$

Using this equality for  $P$  and  $Q$ , using Theorem 2 for  $Q = G$  and  $P$  successively equal to  $P$  and  $Q$ , and our definition of  $\exp_{X_P, X_Q, J}$  one deduces (iv) for all  $Q$ .

It remains to prove (i). One has  $y_P G = x_P G$  if and only if  $x \in \mathcal{X}_M^G$  and  $y \in X_M$  is such that  $x \approx_M y$  (cf. Lemma 1). From the "if part" and the definition above of  $\exp_{X_P, X_Q, J}$ , the image of  $N_{X_Q}(x, P, C)$  by  $\exp_{X_Q, X_P, J}$  is contained in  $x_P G$ . Then the "only if part" implies (i).  $\square$

The following proposition is an easy consequence from the definition in part (ii) of the Theorem above.

**Proposition 2** *With the notation of Theorem, 2, one has*

$$\exp_{X_P, X_Q, J}(\exp_{X_Q, X, J}(xJ)) = \exp_{X_P, X, J}(xJ), x \in N_{X, J}(P, C)$$

The following assertion is an immediate corollary of the Cartan decomposition for  $X_Q$ .

Let  $C > 0$ . The complementary set in  $X_Q$  of the union of  $N_{X_Q}(P, C)$  when  $P$  describes the maximal standard  $\sigma$ -parabolic is a compact set modulo the action of  $A_L$ . (4.9)

## 5 Eisenstein integrals and some results of Nathalie Lagier

### 5.1 Eisenstein integrals

Let  $P = MU$  be a semi standard  $\sigma$ -parabolic subgroup of  $G$ . Let  $(\delta, E)$  be a unitary irreducible smooth representation of  $M$ . Let  $\chi \in X(M)_\sigma$  and let  $\delta_\chi = \delta \otimes \chi$  and let us denote by  $E_\chi$  the space of  $\delta_\chi$ . Let  $(i_P^G \delta_\chi, i_P^G E_\chi)$  be the normalized parabolically induced representation.

The intertwining linear map from  $i_P^G \check{\delta}$  to  $(i_P^G \delta)^\vee$  which associates to  $\check{v} \in i_P^G \check{\delta}$  the linear form on  $i_P^G \delta$  given by the absolutely converging integrals:

$$v \mapsto \int_{U^-} \langle \check{v}(u^-), v(u^-) \rangle du^-, v \in i_P^G \delta \quad (5.1)$$

is an isomorphism.

The restriction of functions to  $K_0$  determines a bijection between  $i_P^G E_\chi$  and  $i_{K_0 \cap P}^{K_0} E$ . If  $v$  is an element of  $i_{K_0 \cap P}^{K_0} E$ ,  $v_\chi$  will denote its unique extension to an element of  $i_P^G E_\chi$ .

$$\text{Let } \mathcal{V}(\delta, H) = \oplus_{x \in \mathcal{X}_M^G} \mathcal{V}(\delta, x, H) \text{ where } \mathcal{V}(\delta, x, H) = (E')^{M \cap x^{-1} \cdot H}. \quad (5.2)$$

Let  $\eta = (\eta_x)_{x \in \mathcal{X}_M^G} \in \mathcal{V}(\delta, H)$ . Let  $J_\chi$  be the subspace of elements of  $i_P^G E_\chi$  whose support is contained in  $P\mathcal{W}_M^G H$  which is the union of the open  $(P, H)$  double cosets in  $G$ . One defines a linear form on  $J_\chi$  by

$$\langle \tilde{\xi}(P, \delta_\chi, \eta), v \rangle = \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1} \cdot H \setminus x^{-1} \cdot H} \langle \eta_x, v(yx^{-1}) \rangle dy, v \in J_\chi.$$

From [BD], Theorem 2.7, one sees that

$$\begin{aligned} &\text{There exists a non zero product } q \text{ of functions on } X(M)_\sigma \text{ of the form} \\ &\chi \mapsto \chi(m) - c, \text{ for some } m \in M \text{ and } c \in \mathbb{C}^*, \text{ such that if } q(\chi) \neq 0, \\ &\tilde{\xi}(P, \delta_\chi, \eta) \text{ extends to a unique } H\text{-invariant linear form on } i_P^G E_\chi, \text{ denoted} \\ &\text{by } \xi(P, \delta_\chi, \eta). \text{ Moreover for every } v \text{ element of } i_{K_0 \cap P}^{K_0} E, \text{ the map } \chi \mapsto \\ &q(\chi) \langle \xi(P, \delta_\chi, \eta), v_\chi \rangle \text{ extends to a polynomial function on } X(M)_\sigma. \end{aligned} \quad (5.3)$$

When  $\xi(P, \delta_\chi, \eta)$  is defined, one defines for  $v \in i_P^G E_\chi$ :

$$E(P, \delta_\chi, \eta, v)(\dot{g}) = \langle \xi(P, \delta_\chi, \eta), (i_P^G \delta_\chi)(g)v \rangle, g \in G.$$

Now, one uses (9.4) which extends results of [BD] and [L] when the characteristic of  $\mathbf{F}$  is equal to zero to the case where this characteristic is different from 2. From (9.4), [L], Theorem 4 (ii), [BD], Theorem 2.14 and Equation (2.33), one sees that if  $\chi \in X(M)_\sigma$  is such that  $Re(\chi \delta_P^{-1/2})$  is strictly  $P$ -dominant,  $\xi(P, \delta_\chi, \eta)$  is defined and one has:

$$E(P, \delta_\chi, \eta, v)(\dot{g}) = \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1} \cdot H \setminus x^{-1} \cdot H} \langle \eta_x, v(yx^{-1}g) \rangle dy, g \in G, v \in i_P^G E_\chi \quad (5.4)$$

the integrals being convergent.

## 5.2 Some results of Nathalie Lagier

One has the following assertion which follows from [W], Theorem IV.1.1. Let  $P = MU, P' = MU'$  be two  $\sigma$ -parabolic subgroups of  $G$  with Levi subgroup  $M$ .

There exists  $R > 0$  such that if  $\chi \in X(M)_\sigma$  satisfies

$$\langle \operatorname{Re}(\chi), \alpha \rangle > R, \alpha \in \Delta(P) \cap \Delta(P'^-),$$

the following integrals are convergent:

(5.5)

$$(A(P', P, \delta_\chi)v)(g) := \int_{U \cap U' \setminus U'} v(u'g) du', v \in i_P^G E_\chi$$

Then  $A(P', P, \delta_\chi)$  is an intertwinning operator between  $i_P^G \delta_\chi$  and  $i_{P'}^G \delta_\chi$ .

The following results are due to Nathalie Lagier (cf. [L], Theorem 5). We use the notation and hypothesis of the preceding subsection.

Let  $P$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . Let  $(a_n)$  be a sequence in  $A_M$  such that  $(a_n) \rightarrow_P \infty$  i.e. such that for every root  $\alpha$  of  $A_M$  in the Lie algebra of  $U$ ,  $(|\alpha(a_n)|_F)$  tends to infinity.

Let  $(\delta, E)$  be a smooth unitary irreducible representation of  $M$  and let  $\mu_\delta$  be its central character. Let  $\chi \in X(M)_\sigma$ . Let us assume that the real part of  $\tilde{\chi} := \chi \delta_P^{-1/2}$  is strictly  $P$ -dominant and satisfies (5.5) for  $P' = P^-$ . Let  $v \in i_P^G E_\chi$  and  $g \in G$ . Recall that we have chosen  $\mathcal{X}_M^G \subset \mathcal{X}_{M_\emptyset}^G$  such that  $\dot{1} \in \mathcal{X}_M^G$ . Then one has:

If  $\eta \in \mathcal{V}(\delta, x, H)$  with  $x \in \mathcal{X}_M^G$  different from  $\dot{1}$ , one has:

(5.6)

$$\lim_{n \rightarrow \infty} \tilde{\chi}(a_n^{-1}) \mu_\delta(a_n^{-1}) E(P, \delta_\chi, \eta, v)(\dot{1} a_n g) = 0.$$

and

If  $\eta \in \mathcal{V}(\delta, 1, H)$ , i.e.  $\eta \in E^{M \cap H}$ , one has the equality of

$$\lim_{n \rightarrow \infty} \tilde{\chi}(a_n^{-1}) \mu_\delta(a_n^{-1}) E(P, \delta_\chi, \eta, v)(\dot{1} a_n g) \quad (5.7)$$

with

$$\langle \eta, (A(P^-, P, \delta_\chi)v)(g) \rangle,$$

Let  $\varepsilon$  be the trivial representation of  $M_\emptyset$ . Let  $\chi \in X(M_\emptyset)_\sigma$  such that the real part of  $\tilde{\chi} := \chi \delta_P^{-1/2}$  is strictly  $P_\emptyset$ -dominant. Let  $\eta$  be the linear form on  $\mathbb{C}$  corresponding to 1 and let  $x \in \mathcal{X}_M^G$ . We consider the Eisenstein integrals for  $x^{-1}.H \setminus G$ . Then  $x^{-1}$  might be viewed as an element of a set  $\mathcal{X}_M^G$  for  $x^{-1}.H$ . We view  $\eta$  as an element of  $E^{M \cap H} = \mathcal{V}(\varepsilon, 1, H)$  and of  $E^{M \cap x x^{-1}.H} = \mathcal{V}(\varepsilon, x^{-1}, x^{-1}.H)$ . Let  $v \in i_{P_\emptyset}^G \chi$ . We denote by  $E_x(P_\emptyset, \eta, v)$  the Eisenstein integral for  $x^{-1}.H \setminus G$ . Then one has:

$$E(P_\emptyset, \chi, \eta, v)(xg) = E_x(P_\emptyset, \chi, \eta, v)((x^{-1}.H)g), g \in G,$$

as it follows easily from (5.4). Using this, it follows from [L], Theorem 6:

Let  $\varepsilon$  be the trivial representation of  $M_\emptyset$ . Let  $\chi \in X(M_\emptyset)_\sigma$  such that the real part of  $\tilde{\chi} := \chi \delta_P^{-1/2}$  is strictly  $P_\emptyset$ -dominant. Let  $P = MU$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . Let  $(a_n)$  be a sequence in  $A_M$  such that  $(a_n) \rightarrow_P \infty$ .

Let  $\eta$  be the linear form on  $\mathbb{C}$  corresponding to 1. Let  $x \in \mathcal{X}_M^G$ . We view  $\eta$  as an element of  $\mathcal{V}(\varepsilon, 1, H)$ . For  $v \in V := i_P^G \mathbb{C}_\chi$ , let  $E_v := E(P_\emptyset, \chi, \eta_x, v)$ . Then the sequence  $(\tilde{\chi}(a_n^{-1})E_v(xa_n))$  has a limit. If  $x \notin \dot{1}P$  this limit is equal to zero. Moreover if  $x = \dot{1}$ , one has:

$$\lim_{n \rightarrow \infty} (\tilde{\chi}(a_n^{-1})E_v(\dot{1}a_n)) = l(v), v \in V$$

where  $l$  is a non zero linear form on  $V$ .

Actually  $l$  is explicit but what is important for us here is that it is non zero.

### 5.3 Applications of the results of N. Lagier

**Lemma 7** *Let  $P = MU$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . Let  $(a_n)$  be a sequence in  $A_M$  such that  $(a_n) \rightarrow_P \infty$ . If  $(g_n)$  is a sequence in  $G$  converging to  $g \in G$  and such that for all  $n \in \mathbb{N}$ ,  $\dot{1}a_n g_n = \dot{1}a_n$ , then  $g$  is an element of  $H_P = U^-(M \cap H)$ .*

*Proof :*

One applies (5.7). We use the notation of this result. If  $J$  is a compact open subgroup of  $G$ , for  $n$  large enough  $g_n J = gJ$ . Hence, if  $v \in i_P^G E_\chi$ ,

$$E(P, \delta_\chi, \eta, v)(\dot{1}a_n g_n) = E(P, \delta_\chi, \eta, v)(\dot{1}a_n g),$$

for  $n$  large enough.

First, let  $\delta$  be the trivial representation of  $M$ . One applies (5.7) to  $v$  and  $(i_P^G \chi(g))v$  in order to deduce from the preceding equality

$$(A(P^-, P, \chi)v)(g) = (A(P^-, P, \chi)v)(1), v \in i_P^G \mathbb{C}_\chi$$

for  $\chi$  sufficiently  $P$ -dominant. If  $\chi$  is such that  $A(P^-, P, \chi)$  is bijective, one deduces the following equality:

$$v(g) = v(1), v \in i_{P^-}^G \mathbb{C}_\chi. \quad (5.9)$$

Let us show that this implies  $g \in U^-M$ . Let us write  $g = p^-k$  with  $k \in K_0$  and  $p^- \in P^-$ . If  $k \notin K_0 \cap P^-$ , there exists  $v \in i_{P^-}^G \mathbb{C}_\chi$  such that  $v(k) = 0$  and  $v(1) = 1$ , as the space of restrictions to  $K_0$  of the elements of  $i_{P^-}^G \mathbb{C}_\chi$  is equal to  $i_{K_0 \cap P^-}^{K_0} \mathbb{C}$ . This is a contradiction to (5.9). Hence  $g = u^-m$  with  $u^- \in U^-$  and  $m \in M$ .

Then applying (5.7) to any  $(\delta, E, \eta)$ , we get similarly the equality:

$$\langle \delta'(m)\eta, e \rangle = \langle \eta, e \rangle, e \in E.$$

The abstract Plancherel formula (cf. [Ber], section 0.2) for  $H \cap M \backslash M$  implies  $m \in M \cap H$ .  $\square$

**Lemma 8** *Let  $P = MU, P' = M'U'$  be two standard  $\sigma$ -parabolic subgroups of  $G$ . Let  $(a_n)$  (resp.,  $(a'_n)$ ) be a sequence in  $A_M$  (resp.  $A_{M'}$ ) such that  $(a_n) \rightarrow_P \infty$  (resp.  $(a'_n) \rightarrow_{P'} \infty$ ). Let  $J$  be a compact open subgroup of  $G$ . Let us assume that there exists  $g, g' \in G$  such that for all  $n \in \mathbb{N}$ ,  $\dot{1}a_n g J = \dot{1}a'_n g' J$ . Then, taking possibly subsequences, one has:*

- (i) *for all  $\chi$  such that the real part of  $\tilde{\chi} := \chi \delta_P^{-1/2}$  is strictly  $P_\emptyset$ -dominant  $\tilde{\chi}(a_n^{-1}a'_n)$  has a non zero limit.*
- (ii) *The sequence  $(a_n^{-1}a'_n)$  is bounded.*
- (iii) *One has  $P = P'$ .*
- (iv) *If  $Q$  is a  $\sigma$ -parabolic subgroup of  $G$  such that  $P \subset Q$ , one has  $\dot{1}_Q a_n g J = \dot{1}_Q a'_n g' J$  for  $n$  large.*

*Proof :*

- (i) For all  $n \in \mathbb{N}$ , there exists  $j_n \in J$  such that

$$\dot{1}a_n g = \dot{1}a'_n g' j_n. \quad (5.10)$$

As  $J$  is compact, one may take a subsequence and we may assume that  $(j_n)$  converges to  $j \in J$ . Let  $g'' = g' j g^{-1}$ . One will apply the result (5.8). With its notations, let  $v \in i_{P_\emptyset}^G \mathbb{C}_\chi$  and let us denote by  $E_v$  the function  $E(P_\emptyset, \chi, \eta, v)$ . As  $E_v$  is right invariant by an open compact subgroup of  $G$ , one has  $E_v(\dot{1}a'_n g' j_n g^{-1}) = E_v(\dot{1}a'_n g'')$  for  $n$  large. From (5.8), one has:

$$\lim_n \tilde{\chi}(a_n^{-1}) E_v(\dot{1}a_n) = l(v), \lim_n \tilde{\chi}(a_n'^{-1}) E_v(\dot{1}a'_n g' j_n g^{-1}) = l'(v) \quad (5.11)$$

where  $l, l'$  are non zero linear forms on  $i_{P_\emptyset}^G E_\chi$ . Also from (5.10) one has:

$$\dot{1}a_n = \dot{1}a'_n g' j_n g^{-1} \quad (5.12)$$

Let us show that there exists  $v_1 \in V = i_{P_\emptyset}^G \mathbb{C}_\chi$  such that  $l(v_1)$  and  $l'(v_1)$  are non zero. Let  $v \in V$  such that  $l(v) \neq 0$ . Then  $l$  does not vanish on  $v + \text{Ker}(l)$ . If  $l'$  vanished identically on  $v + \text{Ker}(l)$  it would vanish on  $V$ , a contradiction which shows that  $l'$  does not vanish identically on  $v + \text{Ker}(l)$ . This proves our claim.

For such a  $v_1$ , one sees from (5.11) and (5.12) that:

$$\text{The sequence } (\tilde{\chi}(a_n a_n'^{-1})) \text{ tends to a non zero limit.} \quad (5.13)$$

This proves (i).

- (ii) By varying  $\chi$  such that  $\chi = \text{Re} \chi$  and such that  $\text{Re} \chi$  describes a basis of  $\mathfrak{a}_\emptyset^*$  one gets (ii).

- (iii) If  $P$  is different from  $P'$ , by exchanging possibly the role of  $P$  and  $P'$ , there exists a simple root  $\alpha$  of  $A_\emptyset$  in the Lie algebra of  $U$  which is not a root in the Lie algebra of  $U'$ , hence which is a root in the Lie algebra of  $M'$ . Then  $|\alpha(a'_n)|_{\mathbf{F}} = 1$  and  $|\alpha(a_n a_n'^{-1})|_{\mathbf{F}}$  is unbounded. This would contradict (ii). Hence  $P = P'$  and (iii) is proved.

- (iv) From (ii), one writes  $a'_n = a_n b_n$  where the sequence  $(b_n)$  in  $A_M$  is bounded. Taking a subsequence we can assume that  $(b_n)$  converges to  $b \in A_M$ .

Taking into account (5.10), one has  $\dot{1}a_n = \dot{1}a_n c_n$  where  $c_n = b_n g' j_n g^{-1}$ . One deduces

from Lemma 7 that the limit  $c$  of  $(c_n)$  is in  $H_P$ . One has  $a'_n g' J = a_n c_n g J$ . Hence for  $n$  large one has:

$$\dot{1}_P a'_n g' J = \dot{1}_P a_n c_n g J = \dot{1}_P a_n c g J$$

As  $c \in H_P$  and as  $a_n \in A_M$  normalize  $H_P$ , one deduces that, for  $n$  large:

$$\dot{1}_P a'_n g' J = \dot{1}_P a_n g J.$$

This proves (iv) for  $Q = P$ .

Let  $g, g' \in G$ . In view of Theorem 1, applied to the right translates of  $f$  by  $g, g'$ , there exists  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  greater than  $N$  and for all  $f \in C^\infty(H_Q \backslash G)$  which is  $J$ -invariant  $(c_{P,Q} f)(\dot{1}_P a_n g) = f(\dot{1}_Q a_n g)$  and  $(c_{P,Q} f)(\dot{1}_P a'_n g') = f(\dot{1}_Q a'_n g')$ . Let  $f$  be the characteristic function of  $\dot{1}_Q a_n g J \subset X_Q$ . Let  $n$  be an integer greater than  $N$  and let  $x = a_n g, x' = a'_n g'$ . From the above remark, one has:

$$(c_{P,Q} f)(\dot{1}_P x) = f(\dot{1}_Q x) = 1,$$

$$(c_{P,Q} f)(\dot{1}_P x') = f(\dot{1}_Q x').$$

From (iv) for  $Q = P$ , one has  $\dot{1}_P x' J = \dot{1}_P x J$ . By  $J$ -invariance, this implies:

$$(c_{P,Q} f)(\dot{1}_P x') = (c_{P,Q} f)(\dot{1}_P x).$$

Hence, one has

$$f(\dot{1}_Q x') = 1$$

and  $x' \in \dot{1}_Q x J$ . This implies  $\dot{1}_Q x J = \dot{1}_Q x' J$ . □

**Lemma 9** *Let  $P = MU, P' = M'U'$  be two standard  $\sigma$ -parabolic subgroups of  $G$ . Let  $(a_n)$  (resp.,  $(a'_n)$ ) be a sequence in  $A_M$  (resp.,  $A_{M'}$ ) such that  $(a_n) \rightarrow_P \infty$  (resp.,  $(a'_n) \rightarrow_{P'} \infty$ ). Let  $g, g' \in G$  and  $x, y \in \mathcal{X}_{M_0}^G$ . Let us assume that the sequences  $(x a_n g J)$  and  $(y a'_n g' J)$  are equal. Then one has  $P = P', xP = yP$  and  $y = xm$  for some  $m \in M$ .*

*Proof :*

Let  $\chi \in X(M_0)_\sigma$  such that  $\chi = |\chi|$  and such that  $\text{Re}(\tilde{\chi})$  is strictly  $P_0$ -dominant. By exchanging possibly the role of  $x$  and  $y$ , and by taking a subsequence, one may assume that  $\tilde{\chi}(a_n) \geq \tilde{\chi}(a'_n)$ . Changing  $H$  into  $x^{-1}H$ , one is reduced to the case where  $x = \dot{1}$ . Using the notation and the result of (5.8), one sees that there exists a non zero linear form  $l$  on  $V_\chi := i_{P_0}^G \mathbb{C}_\chi$  such that for all  $v \in V_\chi$ , one has:

$$\lim_n \tilde{\chi}(a_n)^{-1} E_v(\dot{1} a_n g) = l(v). \quad (5.14)$$

Let  $v \in V_\chi$  such that  $l(v) \neq 0$ . One chooses  $j_n \in J$  such that  $\dot{1} a_n g = y a'_n g' j_n$ . By extracting a subsequence, one may assume that  $(j_n)$  converges to  $j \in J$ . One has:

$$E_v(\dot{1} a_n g) = E_v(y a'_n g' j) \text{ for } n \text{ large} \quad (5.15)$$

and  $\tilde{\chi}(a_n) \geq \tilde{\chi}(a'_n)$ . Let us assume  $y \notin \dot{1} P'$ . Then from (5.8)

$$\lim_n \tilde{\chi}(a'_n)^{-1} E_v(y a'_n g' j) = 0.$$



Together with (5.15) this contradicts (5.14). Hence  $y \in \dot{1}P'$  which implies (cf. (3.3))  $y = \dot{1}m'$  for some  $m' \in M'$ . This implies the equality  $ya'_ng' = \dot{1}a'_nm'g'$  as  $a'_n \in A_{M'}$ . Hence one has  $\dot{1}a_ngJ = \dot{1}a'_nm'g'J$ . Using Lemma 8, one sees that  $P = P'$ . Hence  $M = M'$  and the lemma follows.  $\square$

## 6 End of proof of Theorem 2

### 6.1 Definition of $\exp_{X_P, X, J}$

We have to deal only with the case  $Q = G$  i.e.  $X_Q = X$ . If it does not exist a constant  $C > 0$  satisfying (i) of Theorem 2 for  $Q = G$ , there would exist  $x, y \in \mathcal{X}_{M_\emptyset}^G$ , and sequences  $(a_n), (a'_n) \in A_\emptyset$ ,  $(\omega_n), (\omega'_n) \in \Omega$  such that  $xP \neq yP$ ,  $(|\alpha(a_n)|_{\mathbf{F}})$  tends to infinity for all roots  $\alpha$  of  $A_\emptyset$  in the Lie algebra of  $U$  and such that:

$$xa_n\omega_nJ = ya'_n\omega'_nJ, n \in \mathbb{N}.$$

By extracting subsequences, one may assume that  $\omega_n$  (resp.,  $\omega'_n$ ) converges to  $\omega$  (resp.  $\omega'$ ). Let  $Q = LV$  be the standard  $\sigma$ -parabolic subgroup of  $G$  such that for  $\alpha \in \Delta(P_\emptyset)$ , the sequence  $(|\alpha(a_n)|_{\mathbf{F}})$  is unbounded if and only if  $\alpha \in \Delta(Q, A_\emptyset)$ . Clearly one has  $Q \subset P$ .

By extracting subsequences, one will show that one can write  $a_n = b_nc_n$  where the sequence  $(b_n)$  in  $A_L$  satisfies  $(b_n) \rightarrow_Q \infty$  and where the sequence  $(c_n)$  converges in  $G$ . Let  $(\delta_1, \dots, \delta_p)$  be the union of  $\Delta(Q, A_\emptyset)$  viewed as subset of  $\mathfrak{a}'_\emptyset$  and of a basis of  $\mathfrak{a}'_G$  viewed as a subset of  $\mathfrak{a}'_\emptyset$  (cf. (2.9)). Let us look to the map  $\phi : A_\emptyset \rightarrow \mathbb{R}^p$  given by  $a \mapsto (\delta_1(H_\emptyset(a)), \dots, \delta_p(H_\emptyset(a)))$ . Its image is a lattice of dimension  $p$  as the image  $\mathfrak{a}_{\emptyset, \mathbf{F}}$  of  $A_\emptyset$  by  $H_\emptyset$  is a lattice of dimension equal to the dimension of  $\mathfrak{a}_\emptyset$ . Its restriction to  $A_L$  has the same property as it factors through  $H_L$  and  $(\delta_1, \dots, \delta_p)$  might be viewed as a basis of  $\mathfrak{a}'_L$ . Hence  $\phi(A_L)$  is of finite index in  $\phi(A_\emptyset)$ . Hence one can find  $x_1, \dots, x_q \in A_\emptyset$  such that for all  $a \in A_\emptyset$  there exists  $b \in A_L$  and  $i \in \{1, \dots, q\}$  such that  $\phi(a) = \phi(bx_i)$ . This allows to define  $b_n$  and  $c_n = a_n(b_n)^{-1}$ . One has  $c_n = x_{i_n}$  for some  $i_n \in \{1, \dots, q\}$ . Then extracting a subsequence one may even assume that  $(c_n)$  is constant hence it converges. Moreover as  $\phi(b_n) = \phi(a_n) - \phi(x_{i_n})$  one has  $(b_n) \rightarrow_Q \infty$ .

Hence, for  $n$  large,  $xa_n\omega_nJ = xb_nc\omega J$  where  $c$  is the limit of  $(c_n)$ . We introduce similarly  $Q'$ ,  $b'_n$  and  $c'_n$ . From Lemma 9 applied to  $G$  one deduces  $Q' = Q$  and  $HxQ = HyQ$ . Hence, as  $Q \subset P$ , one has  $xP = yP$ . A contradiction which shows that there exists  $C > 0$  which satisfies (i). It is clear that any constant greater than such a constant enjoys the same property.

Let us assume that there is no constant satisfying (i) which satisfies also (ii). Proceeding as above, there would exist sequences  $(b_n)$  in  $A_L$ ,  $(b'_n)$  in  $A_L$ ,  $c, c' \in G$ , two standard  $\sigma$ -parabolic subgroups  $Q = LV, Q' = L'V' \subset P$  of  $G$  and  $x, y \in \mathcal{W}_{M_\emptyset}^G$  such that,  $(b_n) \rightarrow_Q \infty$ ,  $(b'_n) \rightarrow_{Q'} \infty$ , and

$$\begin{aligned} xb_ncJ &= yb'_ncJ. \\ x_P b_ncJ &\neq y_P b'_ncJ. \end{aligned} \tag{6.1}$$

From Lemma 9, one sees that  $Q = Q'$  and  $x \approx_L y$ . In particular  $y = xl$  for some  $l \in L$  and, as  $l$  commutes to the elements  $b'_n$  of  $A_L$ , one has:

$$xb_n cJ = xb'_n l c' J.$$

Conjugating  $x^{-1}$ , one gets an equality of left  $x^{-1}.H$  cosets. From Lemma 8 (i), applied to  $x^{-1}.H$  instead of  $H$ , one deduces that  $(b'_n b_n^{-1})$  is bounded. Hence, by taking a subsequence one can assume that it has a limit. Then from Lemma 8 (iii) one gets for  $n$  large:

$$(x^{-1}.H)_P b_n cJ = (x^{-1}.H)_P b'_n l c' J.$$

Hence there exists a sequence in  $J$ ,  $(j_n)$  such that

$$(x^{-1}.H)_P b_n c j_n = (x^{-1}.H)_P b'_n l c'.$$

Hence  $b_n c j_n c'^{-1} l^{-1} b'_n \in (x^{-1}.H)_P$ . As the stabilizer of  $x_P$  is equal to  $(x^{-1}.H)_P$ , one deduces from this the equality:

$$x_P b_n c j_n = x_P b'_n l c$$

As  $x = yl$  and  $l \in L \subset M$ , one has  $x_P = y_P l$ . As  $l \in L$  commutes to  $b'_n \in A_L$ , one deduces from this the equality

$$x_P b_n cJ = y_P b'_n c' J,$$

for  $n$  large. This contradicts our hypothesis (6.1). Hence there exists  $C > 0$  which satisfies (i) and (ii).

## 6.2 Injectivity of $\exp_{X_P, X, J}$

Let us prove that one can choose  $C > 0$  such that  $\exp_{X_P, X}$  is injective on  $N_{X_Q, J}(P, C)$ . Let us assume that every constant  $C > 0$  satisfying conditions (i), (ii) of Theorem 2 does not satisfy condition (iii). From the finiteness of  $\mathcal{X}_{M_\emptyset}^G$  and proceeding as in section 6.1, one sees that there would exist  $x, x' \in \mathcal{X}_{M_\emptyset}^G$ , two  $\sigma$ -parabolic subgroups  $Q = LV, Q' = L'V' \subset P$  of  $G$ , a sequence  $(a_n)$  in  $A_L$ , a sequence  $(a'_n)$  in  $A_{L'}$  such that  $(a_n) \rightarrow_Q \infty$ ,  $(a'_n) \rightarrow_{Q'} \infty$  and two elements  $d$  and  $d'$  of  $A_\emptyset \Omega$  such that:

$$xa_n dJ \neq x'a'_n d' J$$

and

$$x_P a_n dJ = x'_P a'_n d' J.$$

Let  $f_n$  be the characteristic function of  $xa_n dJ$ . For  $n_0$  large enough one can use Theorem 1 (iv) for the right translates of  $f_{n_0}$  by  $d$  and  $d'$  and one has, by setting  $a = a_{n_0}$ ,  $f = f_{n_0}$ , etc.:

$$f(xad) = (c_{P, G} f)(x_P a d), f(x'a'd') = (c_{P, G} f)(x'_P a' d')$$

But, by our assumptions  $f(xad) = 1$  and  $x_P a dJ = x'_P a' d' J$ . Hence, by  $J$  invariance, one has:

$$f(x'a'd') = 1$$

which implies

$$xadJ = x'a'd'J.$$

This is a contradiction to our hypothesis. This achieves to prove that there exists a constant  $C > 0$  such that the properties (i), (ii) and (iii) of Theorem 2 are satisfied.

### 6.3 Volumes

The following lemma will allow to finish the proof of Theorem 2.

**Lemma 10** *Let  $K$  be a compact open subgroup of  $G$  with a strong  $\sigma$ -factorization for standard  $\sigma$ -parabolic subgroups (cf. (4.8)). Let  $P = MU$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . Let  $a \in A_\emptyset$  which is  $P_\emptyset$ -dominant. Then*

(i)

$$HaK = HaK_MK_U.$$

where  $K_M = K \cap M, K_U = K \cap U$ .

(ii)

$$\text{vol}_X \mathbf{i}aK = \text{vol}_{X_P} \mathbf{i}_PaK.$$

*Proof :*

(i) As  $K_{M_\emptyset}K_{U_\emptyset} = K \cap P_\emptyset$  and  $K_MK_U = K \cap P$ , it is enough to prove (i) when  $P = P_\emptyset$ . Let us assume this in the sequel. If  $u^- \in K_{U^-}$ , as  $a$  is  $P_\emptyset$ -dominant, one has  $a.u^- = au^-a^{-1} \in K_{U^-} \subset K = K_HK_MK_U$  (cf. (4.8) (ii) and (iii)). Hence one has:

$$Hau^- = H(a.u^-)a \in HK_MK_Ua.$$

But  $K_MK_Ua = a(a^{-1}.K_M)(a^{-1}.K_U)$ . As  $M = M_\emptyset$  and  $a \in A_\emptyset$ ,  $a^{-1}.K_M = K_M$ . As  $a$  is  $P_\emptyset$ -dominant  $a^{-1}.K_U \subset K_U$  (cf. (4.8) (ii)). Altogether, this shows:

$$HaK_{U^-} \subset HaK_MK_U$$

One deduces (i) from the equality  $K = K_{U^-}K_MK_U$ .

Let us prove (ii). Let  $P$  be a standard  $\sigma$ -parabolic subgroup of  $G$ . As  $U^- \subset H_P$  and  $K = K_{U^-}K_MK_U$ , and  $a.K_{U^-} \subset U^-$ , (cf. (4.8) (ii) ) one has:

$$\mathbf{i}_PaK = \mathbf{i}_PaK_MK_U.$$

Then (ii) follows from (i), from the fact that  $aK_MK_U \subset P$  and from our choice of measure on  $X_P$  (cf. section 4.1).  $\square$

*End of proof of Theorem 2*

Let  $K$  and  $J$  as in the theorem. The proof of (iv) reduces to prove the statement for subsets of  $N(x, P, C)/J$  for  $x \in \mathcal{X}_M^G$ . Using our choices of volumes and translating sets on the left by  $x^{-1}$  and changing  $H$  in  $x^{-1}.H$ , one is reduced to the case  $x = 1$ . For  $K$  and  $J$  as in the theorem, we have:

$$\omega.J \subset K, \omega \in \Omega.$$

Let  $\omega \in \Omega$  and one sets  $J' := \omega.J \subset K$ . As  $\Omega$  is compact and is left  $K$ -invariant,  $\Omega/J$  is finite and  $J'$  varies in a finite set. Let us assume that  $C > 0$  satisfies Theorem 2 (i), (ii) and (iii) for all groups  $J'$ . One has to prove that for  $a \in A_\emptyset^+(P, C)$ :

$$\text{vol}_X(\dot{1}a\omega J) = \text{vol}_{X_P}(\dot{1}_Pa\omega J).$$

As the measures on  $X$  and  $X_P$  are right invariant by  $G$ , in order to prove this equality, it is enough to prove the equality:

$$\text{vol}_X(\dot{1}aJ') = \text{vol}_{X_P}(\dot{1}_PaJ').$$

Let  $K_a$  (resp.,  $K'_a$ ) be the stabilizer in  $K$  of  $\dot{1}a$  (resp.,  $\dot{1}_Pa$ ). We need the following fact. Let  $K_1$  be a closed subgroup of  $K$ . Let us assume that a Haar measure is given on  $K$  and let  $K_1 \backslash K$  be endowed with the image of this measure. Let  $X \subset K$  and  $Y$  its image in  $K_1 \backslash K$ . Then  $\text{vol}_{K_1 \backslash K}(Y) = \text{vol}_K(K_1 X)$ . From this applied to  $K_1 = K_a$  and  $K_1 = K'_a$  and from Lemma 10 (ii), it is enough to prove the equality:

$$K_a J' = K'_a J'.$$

The image of the set  $\dot{1}aK'_a J'$  by the map  $\exp_{X_P, X, J'}$  is equal  $\dot{1}_PaJ'$ , as it follows from the definition in Theorem 2 and the equality  $\dot{1}_PaK'_a J' = \dot{1}_PaJ'$ . From the definition of  $\exp_{X_P, X, J'}$ , this image is also equal to the image of  $\dot{1}aJ'$ . Hence from the part (iii) of Theorem 2, one deduces the equality:

$$\dot{1}aJ' = \dot{1}aK'_a J'$$

Looking to the orbit of  $\dot{1}a$  under  $K$  one deduces from this the inclusion:

$$K'_a J' \subset K_a J'.$$

We recall that  $K \subset \Omega$ . To prove the reverse inclusion let us remark that  $\dot{1}aK_a J'$  is equal to  $\dot{1}aJ'$ . From the definition of  $\exp_{X_P, X, J'}$  one deduces the equality:

$$\dot{1}_PaK_a J' = \dot{1}_PaJ'$$

which implies as above:

$$K_a J' \subset K'_a J'.$$

This implies the required equality. This finishes the proof of the theorem.  $\square$

## 7 Bernstein maps and Scattering Theorem

### 7.1 Constant term and $\exp$ -mappings

The following proposition is an immediate corollary of Theorems 1 and 2.

**Proposition 3** *Let  $P \subset Q$  be two standard  $\sigma$ -parabolic subgroups of  $G$ . Let  $J$  be a compact open subgroup of  $G$  small enough to satisfy the conditions of Theorem 2. There exists  $C > 0$  such that  $\exp_{X_P, X_Q, J}$  is well defined on  $N_{X_Q, J}(P, C)$  and satisfies for all  $J$ -invariant function  $f$  on  $X_Q$ :*

$$(c_{P, Q} f)(\exp_{P, Q, J}(xJ)) = f(xJ), xJ \in N_{X_Q, J}(P, C).$$

**Remark 2** *In [SV], for  $G$ -split and  $X$  spherical, the  $\exp$ -mappings are introduced before the maps  $c_{P, Q}$ , by means of wonderful compactifications, and the maps  $c_{P, Q}$  are defined by the relation above.*

## 7.2 Bernstein maps $e_{Q, P}$

We thank Joseph Bernstein for having suggested to us the proof of the following Theorem.

**Theorem 3** *Let  $P = MU \subset Q = LV$  two standard  $\sigma$ -parabolic subgroups of  $G$ . The right  $G$ -invariant measure on  $X_P$  allows to identify  $C_c^\infty(X_P)$  to a subspace of the dual of  $C^\infty(X_P)$ . Let  $e_{Q, P}$  be the restriction of the transpose map of  $c_{P, Q}$  to  $C_c^\infty(X_P)$ . Let  $J$  and let  $C > 0$  be as in Theorem 2 .*

- (i) *Let  $xJ \in N_{X_Q, J}(P, C)$  and  $y = \exp_{X_P, X_Q, J}(xJ)$ . Then the image by  $e_{Q, P}$  of the characteristic function of  $yJ \subset X_P$  is the characteristic function of  $xJ \subset X_Q$ .*
- (ii) *For  $f \in C_c^\infty(X_P)$  supported in  $\exp_{X_P, X_Q, J}(N_{X_Q, J}(P, C))$ ,  $e_{Q, P} f$  has its support in  $N_{X_Q, J}(P, C)$  and*

$$(e_{Q, P} f)(xJ) = f(\exp_{X_P, X_Q}(xJ)), xJ \in N_{X_Q, J}(P, C).$$

- (iii) *The map  $e_{Q, P}$  has its image in  $C_c^\infty(X_Q)^J$ .*

*Proof :*

- (i) We fix a compact open subgroup  $J$  and  $C$  as in the preceding proposition from which we use the notations. Let  $xJ \in N_{X_Q, J}(P, C) \subset X_Q/J$ . Let  $f$  be the characteristic function of  $\exp_{X_P, X_Q, J}(xJ)$  which is a  $J$ -invariant function on  $X_P$ . Let  $g \in C^\infty(X_Q)$ . One has

$$\langle e_{Q, P} f, g \rangle = \langle f, c_{P, Q} g \rangle$$

and by the preceding proposition one sees:

$$\langle e_{Q, P} f, g \rangle = g(xJ).$$

This implies that  $e_{Q, P} f$  is the characteristic function of  $xJ$ . This proves (i).

- (ii) follows by linear combinations.

- (iii) Let  $a \in A_M$  be strictly  $P$ -dominant. Let  $y \in X_P$ . From the Cartan decomposition for  $X_P$  one sees that for  $n$  large,  $a^n y$  is of the form  $a^n y = \exp_{X_P, X_Q, J}(x_n J) \in X_P/J$  for some  $x_n J \in N_{X_Q, J}(P, C) \subset X_Q/J$ .

For  $n \in \mathbb{Z}$ , let  $f_n$  be the characteristic function of  $a^n y J \subset X_P$ . One has just seen that for  $n$  large in  $\mathbb{N}$ ,  $e_{Q,P}(f_n)$  is in  $C_c^\infty(X)^J$ . Let us assume that it is not true for all  $n \in \mathbb{N}$ . Then there would exist  $N \in \mathbb{N}$  such that  $e_{Q,P}(f_n) \in C_c^\infty(X)^J$  for  $n > N$  and such that  $e_{Q,P}(f_N) \notin C_c^\infty(X)^J$ .

We want to apply Theorem A of [AAG] in order to prove that the  $C_c^\infty(G)^J$ -module  $C_c^\infty(X_P)^J$  is finitely generated. For this it is necessary to see that one may apply it to each homogeneous space  $x_P G$  which is isomorphic to  $U^-(M \cap x^{-1}.H) \backslash G$ . The first thing to prove is that for each parabolic subgroup  $R$  of  $G$ , the number of  $(U^-(M \cap x^{-1}.H), R)$ -double cosets is finite. By using conjugacy, one can assume that  $R$  contains  $A_0$ . By the Bruhat decomposition, one has  $G = \cup_i P x_i R$ , where  $(x_i)$  is a finite family of elements of  $G$  normalizing  $A_0$ . It is enough, to prove our claim, to show that for each  $i$ ,  $R_i := (x_i.R) \cap M$  has a finite number of orbits in the symmetric space  $(M \cap x^{-1}.H) \backslash M$ . But  $R_i$  is a parabolic subgroup of  $L$  and our claim follows from [HW], Corollary 6.16.

The second thing to prove, in order to apply Theorem A of [AAG] is that :

For each finite length smooth  $G$ -module  $V$ , the dimension of the space  $V^{U^-(M \cap x^{-1}.H)}$  is finite. (7.1)

But this dimension is precisely the dimension of  $j(V)^{M \cap x^{-1}.H}$  where  $j(V)$  is the Jacquet module of  $V$  with respect to  $P^-$ . This space is finite dimensional (cf. [D], Theorem 4.4.)

Now, one can apply Theorem A of [AAG] to conclude that the  $C_c^\infty(G)^J$ -module  $C_c^\infty(X_P)^J$  is finitely generated. Moreover the algebra  $C_c^\infty(G)^J$  is Noetherian (cf. [R] Corollary of Theorem VI.10.4).

Hence, it follows that an ascending chain of  $C_c^\infty(G)^J$ -submodules of  $C_c^\infty(X_P)^J$  is stationnary.

We apply this to the  $C_c^\infty(G)^J$ -submodules of  $C_c^\infty(X_P)^J$ ,  $M_n$ , generated by  $f_0, \dots, f_{-n}$ . Hence there exists  $n \in \mathbb{N}$  and  $\phi_0, \dots, \phi_n \in C_c^\infty(G)^J$  such that:

$$f_{-n-1} = f_0 * \phi_0 + \dots + f_{-n} * \phi_n$$

Using that the right  $G$ -action and the left  $A_M$ -action commute (cf. Definition 2) and applying the left action of  $a^{n+1+N}$  to the above identity, one gets:

$$f_N = f_{n+1+N} * \phi_0 + \dots + f_{1+N} * \phi_n$$

From Theorem 1,  $c_{P,Q}$  is a morphism of  $G$ -modules. Hence it is also the case for  $e_{Q,P}$ . Hence  $e_{Q,P}(f_N)$  is in  $C_c^\infty(X_Q)^J$ . From the definition of  $N$ , we get a contradiction. Hence in particular,  $e_{Q,P}f$  is in  $C_c^\infty(X_Q)^J$ . The theorem follows by linearity.  $\square$

Let  $(\pi, V)$  be a smooth representation of a parabolic subgroup  $P = MU$  of  $G$ . One denotes by  $(\pi_P, V_P)$  the tensor product of the quotient of  $V$  by the  $M$ -submodule generated by the  $\pi(u)v - v$ ,  $u \in U, v \in V$ , with the representation of  $M$  on  $\mathbb{C}$  given by  $\delta_P^{-1/2}$ . We call it the normalized Jacquet module of  $V$  along  $P$ . We denote the natural projection map from  $V$  to  $V_P$  by  $j_P$  and sometimes  $\pi_P$  will be denoted  $j_P(\pi)$ .

**Lemma 11** *Let  $P$  be a semistandard  $\sigma$ -parabolic subgroup of  $G$ .*

*(i) If  $f \in C_c^\infty(X)$  has its support in  $X_MP$  we define, using (4.5),  $f^P \in C^\infty(X_M)$  by*

$$f^P(xm) = \delta_P^{1/2}(m) \int_U f(xmu) du, x \in \mathcal{X}_M^G, m \in M$$

*Then  $f^P \in C_c^\infty(X_M)$ .*

*(ii) The map  $f \mapsto f^P$  goes through the quotient to an intertwining map between the normalized Jacquet module  $C_c^\infty(X_MP)_P$  of the  $P$ -module  $C_c^\infty(X_MP)$  and  $C_c^\infty(X_M)$ .*

*(iii) This intertwining map is bijective and its inverse define an intertwining injective map  $m_P^X : C_c^\infty(X_M) \rightarrow C_c^\infty(X)_P$ .*

*(iv) One can replace  $X$  by  $X_P$  in (i), (ii) and (iii) and one gets an injective intertwining map  $m_P : C_c^\infty(X_M) \rightarrow C_c^\infty(X_P)_P$*

*Proof :*

(i) follows easily from the definition.

(ii) It is clear that our map goes through the quotient to a map between the normalized Jacquet module  $C_c^\infty(X_MP)_P$  of the  $P$ -module  $C_c^\infty(X_MP)$ . On the other hand, for  $f \in C_c^\infty(X_MP)$  one has:

$$(\rho(m_0)f)^P(xm) = \delta_P^{1/2}(m) \int_U f(xmm_0m_0^{-1}um_0) du$$

One makes the change of variable  $u' = m_0^{-1}um_0$  to achieve to prove the intertwining property of (ii).

As an  $U$ -space,  $X_MP$  is isomorphic to  $X_M \times U$  where  $U$  acts trivially on the first factor. This implies easily (iii).

(iv) is proved similarly. □

**Proposition 4** *We denote by  $j_P(e_P)$  the map between the normalized Jacquet modules  $C_c^\infty(X_P)_P$  and  $C_c^\infty(X)_P$  determined by  $e_P := e_{G,P}$ . Then*

$$j_P(e_P) \circ m_P = m_P^X.$$

*Proof :*

One has to prove;

$$j_P(e_P)(m_P(f)) = m_P^X(f) \tag{7.2}$$

for all  $f \in C_c^\infty(xM)$  and  $x \in \mathcal{X}_M^G$ . Changing  $H$  to  $x^{-1}.H$ , one is reduced to prove (7.2) for  $x = 1$ . One writes the Cartan decomposition for  $M \cap H \backslash M$ :

$$M \cap H \backslash M = \cup_{x \in \mathcal{X}_{M_\emptyset}^M} xA_\emptyset^+(P_\emptyset, P, 0)\Omega_M,$$

where  $\Omega_M$  is a compact set of  $M$  and  $\mathcal{X}_{M_\emptyset}^M$  is the analog of  $\mathcal{X}_{M_\emptyset}^G$ . The  $M$ -module of compactly supported smooth functions on  $M$  is the linear span the characteristic functions of  $ax\omega J$  where  $J$  describes a basis of neighborhood of 1 in  $M$  made of compact open subgroup of  $M$ ,  $x \in \mathcal{X}_{M_\emptyset}^M$ ,  $\omega \in \Omega_M$ ,  $a \in A_\emptyset^+(P_\emptyset, P, 0)$ . As  $m_P, m_P^X, e_P$

are  $M$ -equivariant, one has to prove (7.2) for every  $f$  among a set of generators of this  $M$ -module. Again we reduce to  $x = 1$ . Taking into account (3.8), one can write  $a = a'b$  with  $a' \in A_\emptyset^+$  and  $b \in A_M$ . As  $b$  commutes to  $J$ , one is reduced to prove (7.2) for the characteristic functions of  $\dot{1}a\omega J$ , with  $a \in A_\emptyset^+$  and  $\omega \in \Omega_M$ .

As  $\omega J = \omega J\omega^{-1}\omega$ , the characteristic functions of  $\dot{1}aJ'$  where  $J'$  describes the set of  $\omega.J$  for  $J$  as above,  $a \in A_\emptyset^+$ ,  $\omega \in \Omega_M$  is a set of generators of  $C_c^\infty(\dot{1}M)$ .

Let  $(J'_n)$  be as in Lemma 6. By continuity and compacity, there exists a neighborhood  $\mathcal{V}$  of 1 in  $M$  such that:

$$\omega.\mathcal{V} \subset (J'_0)_M, \omega \in \Omega_M$$

One can assume that all the groups  $J$  above are contained in  $\mathcal{V}$ . Hence all the groups  $J'$  are contained in  $(J'_0)_M$ . For such a group, let  $n \in \mathbb{N}$  such that  $(J'_n)_M \subset J'$ . Then as  $J'$  is the disjoint union of the left  $(J'_n)_M$ -cosets, the characteristic function of  $\dot{1}aJ'$  is a linear combination of the characteristic functions of  $\dot{1}aj'(J'_n)_M$  where  $j'$  describes  $J'$ . But as  $J'_n$  is normal in  $J'_0$  (cf. Lemma 6) and  $J' \subset (J'_0)_M$ ,  $(J'_n)_M$  is normal in  $J'$ . Hence  $\dot{1}aj'J'_n = \dot{1}aJ'_nj'$ . Hence, again by  $M$ -equivariance, one has to prove (7.2) for  $f$  equal to the characteristic function of  $\dot{1}a(J'_n)_M$ ,  $n \in \mathbb{N}$ ,  $a \in A_\emptyset^+$ .

For simplicity we write  $J$  instead of  $J'_n$  and let  $g = \text{vol}(J_U)\delta_P(a)^{1/2}\dot{1}_{aJ_M}$  and let  $f = \dot{1}_{\dot{1}PaJ_MJ_U} \in C_c^\infty(X_P)$ . Then  $f^P = g$ . Then, by definition of  $m_P$ , one has:

$$m_P(g) = j_P(f)$$

where  $j_P(f)$  is the image of  $f$  in the normalized Jacquet module of  $C_c^\infty(X_P)$ . Similarly the characteristic function  $h$  of  $\dot{1}aJ_MJ_U$  satisfies  $h^P = g$ . Hence one has:

$$m_P^X(g) = j_P(h)$$

and

$$(j_P(e_P))(m_P(g)) = j_P(e_P(f)).$$

It remains to prove:

$$(j_P(e_P))(m_P(g)) = m_P^X(g)$$

i.e.

$$(j_P(e_P))(j_P(f)) = j_P(h)$$

For this, it is enough to prove:

$$e_P(f) = h.$$

One has

$$\dot{1}aJ_MJ_U = \dot{1}aJ$$

from Lemma 10. As  $J_{U-}$  is normalized by  $a \in A_\emptyset^+$  (cf. Lemma 6), one has

$$\dot{1}PaJ_MJ_U = \dot{1}PaJ$$

Then the required equality follows from Theorem 3 (i). □



### 7.3 Discrete spectrum

An irreducible subrepresentation of  $C^\infty(X)$ ,  $(\pi, V)$ , is said discrete if the action of  $A_G$  is unitary and the elements of  $V$  are square integrable mod  $A_G$ . Obviously if  $\psi$  is element of the group  $X(G)_{\sigma,u}$  of unitary elements of  $X(G)_\sigma$ , the representation  $\pi_\psi$  of  $G$  in the space  $V_\psi := \{\psi v | v \in V\}$  is also a discrete series. Moreover  $\pi_\psi$  is isomorphic to  $\pi \otimes \psi$ . Let  $\chi$  be a unitary character of  $A_G$  and let  $L^2(X, \chi)_{disc}$  the sum of all  $X$ -discrete series on which  $A_G$  acts by  $\chi$ .

**Theorem 4** *Let  $J$  be a compact open subgroup of  $G$  and  $\chi$  a unitary character of  $A_G$ . Then the space  $L^2(X, \chi)_{disc}^J$  of  $J$ -invariants of  $L^2(X, \chi)_{disc}$  is finite dimensional.*

*Proof :*

One will see that the proof of Theorem 9.2.1 of [SV] adapts by changing  $Z(G)^0$  to  $A_G$ , and, for a standard  $\sigma$ -parabolic subgroup  $P = MU$  of  $G$  by changing  $Z(X_P)$  to  $A_M$  acting on the left. .

Let  $A_P^+$  be the set of  $P$ -dominant elements of  $A_P$ . Let  $N'_P$  be equal to  $N_{X,J}(P, C)$  for  $C > 0$  large enough in such a way that the  $exp$ -maps are defined and such that the identity of Proposition 3 holds. Let  $N_P = N'_P \setminus_{Q \subset P, Q \in \mathcal{P}, Q \neq P} N'_Q$ . Then the  $N_P$  covers  $X$ . We remark that  $exp_{X_P, X, J}(N'_P)$  is stable by the left action of  $A_P^+$  as well as  $N''_P := exp_{X_P, X, J}(N_P)$ . One sees from the definitions that there is a finite subset  $\Omega_P$  of  $X_P/J$ , such that  $N''_P = A_P^+ \Omega_P$ . Let  $(\hat{A})_{\mathbb{C}}^{J_M}$  be the set of complex characters of  $A_M$  which are trivial on  $A_M \cap J$ . Let  $\mathcal{P}$  be the set of standard  $\sigma$ -parabolic subgroups of  $G$ . We choose a map  $R : \mathcal{P} \rightarrow \mathbb{N}$ ,  $P \mapsto r_P$  and we define  $\mathfrak{S}_R := \prod_{P \in \mathcal{P}} ((\hat{A})_{\mathbb{C}}^{J_M})^{r_P}$ . An element of  $x \in \mathfrak{S}_R$  is denoted  $[(\chi_i)_{i=1, \dots, r_P}]_{P \in \mathcal{P}}$ . We consider for  $a \in A_M$ ,

$$\prod_{i=1, \dots, r_P} (\mathcal{L}_a - \chi_i(a)) \quad (7.3)$$

Let  $x \in \mathfrak{S}_R$ . We consider the subspace  $V_x \subset C^\infty(X)^J$  of  $J$ -invariant functions on  $X$ ,  $f$ , such that for all standard  $\sigma$ -parabolic subgroup  $P$  of  $G$  and  $a \in A_M$ ,  $c_{P,G}f$  is annihilated by (7.3). Then  $V_x$  is invariant by the Hecke algebra of  $C_c^\infty$  functions on  $G$  which are right and left invariant by  $G$ : this is due to the fact that  $c_{P,G}$  is a  $G$ -morphism and that the right action of  $G$  on  $C^\infty(X_P)$  commutes with the left action of  $A_P$ .

Recall that from our hypothesis on  $C$  that:

$$(c_{P,G}f)(exp_{X_P, X, J}(x)) = f(x), x \in N_P.$$

Then  $V_x$  is finite dimensional, as it is shown in the proof of Theorem 9.2.1 of [SV]. The rest of the proof is entirely analogous to the proof of this Theorem.  $\square$

**Corollary 1** *Let  $J$  be a compact open subgroup of  $G$ . There exists finitely many discrete series for  $X$ ,  $(\pi_i, V_i)$ ,  $i = 1, \dots, n$  such that any discrete series,  $(\pi, V)$  for  $X$  is of the form  $(\pi_i)_\chi$  where  $\chi$  is element of the group  $X(G)_{\sigma,u}$  of unitary elements of  $X(G)_\sigma$  and  $i \in \{1, \dots, n\}$ .*

*Proof :*

Looking to Lie algebras one sees that restriction map from the group  $X(G)_{\sigma,u}$  of unitary elements of  $X(G)_\sigma$  to the group  $X(A_G)_u$  of unitary elements of  $X(A_G)$  is surjective. On the other hand the action by multiplication of  $X(A_G)_u$  on  $(\hat{A}_G)_u^J$  has finitely many orbits (cf. 2.6). Hence one is reduced to the case where the restriction of the central character of  $\pi$  is one of the representatives of these orbits. Then the corollary follows immediately from the Theorem.  $\square$

The proof of the following Lemma is immediate.

**Lemma 12** *Let  $\delta_{P,\mathcal{X}_M^G}$  be the function on  $X_M$  such that, for all  $x \in \mathcal{W}_M^G$ , its restriction to  $xM$  is equal to the function  $\delta_{P,x}$  occurring in (4.6). For a function  $f$  on  $X_P$  we associate the map  $T(f)$  on  $G$  with values in the space of functions on  $X_M$  defined by:*

$$(T(f)(g))(x) = \delta_{P,\mathcal{W}_M^G}^{-1/2}(x)f(xg), x \in X_M, g \in G$$

(i) *One has*

$$T(f)(mg) = (\rho \otimes \delta_P^{1/2})(m)f(g), m \in M, g \in G.$$

(ii) *The map  $T$  induces a bijective  $G$ -intertwining map between  $C_c^\infty(X_P)$  and  $i_{P^-}^G C_c^\infty(X_M)$  (resp.,  $C^\infty(X_P)$  and  $i_{P^-}^G C^\infty(X_M)$ ).*

(iii) *Let  $\chi$  be a unitary character of  $A_M$ . The map  $T$  induces a bijective isometric  $G$ -intertwining map between  $L^2(X_P)$  and the unitarily induced representation from  $P^-$  to  $G$  of  $L^2(X_M)$  (resp.,  $L^2(X_P, \chi)_{disc}$  and the unitarily induced representation from  $P^-$  to  $G$  of  $L^2(X_M, \chi)_{disc}$ ).*

*Proof :*

(i) is immediate.

(ii) From (i), it remains only to prove the bijectivity. The inverse map to  $T$  is easily described using the fact that  $X_P = X_M \times_{P^-} G$ .

(iii) follows easily from the definition of the scalar product on unitary induced representations from  $P$  to  $G$  (cf. (5.1)) and from the definition of the  $M$ -invariant measure on  $X_M$  (cf. (4.6) and (4.3)).  $\square$

**Lemma 13**  *$L^2(X_P)_{disc}$  is unitarily equivalent to the unitary induced representation from  $P^-$  to  $G$  of  $(L^2(X_M)_{disc})$*

*Proof :*

The Lemma follows from the analog of Corollary 9.3.4 in [SV] and of Lemma 12 (iii). Notice that this Corollary follows from l.c. Equation (9.1). To establish its analog, one remarks that  $A_M$  acts freely on the left on  $X_P$ .  $\square$

**Lemma 14** *The  $G$ -space  $X_P$  satisfies the discrete series conjecture 9.4.6 of [SV] for the parabolic subgroup  $P^-$  and the torus of unitary unramified characters of  $P^-$ ,  $D^* := X(M)_{\sigma,u}$ .*

*Proof :*

From Corollary 1 of Theorem 4, there is a denumerable family of  $X(M)_{\sigma,u}$ -orbits of discrete series. Then the Lemma follows from Lemma 13.  $\square$

## 7.4 Bernstein maps

The proof of the following theorem is entirely analogous to the proof of Theorem 11.1.2 in [SV].

**Theorem 5** *For every pair of standard  $\sigma$ -parabolic subgroups of  $G$ ,  $P \subset Q$ , there exists a canonical  $G$ -equivariant map  $i_{P,Q} : L^2(X_P) \rightarrow L^2(X_Q)$  characterized by the property that for any  $\Psi \in C_c^\infty(X_P)$  and any  $a$  element of the set  $A_P^{++}$  of strictly  $P$ -dominant elements of  $A_P$ , we have:*

$$\lim_{n \rightarrow \infty} (i_{Q,P} \mathcal{L}_{a^n} \Psi - e_{Q,P} \mathcal{L}_{a^n} \Psi) = 0$$

where the limit is in  $L^2(X_Q)$ .

Then as a corollary of Theorem 5 and of the analog of Proposition 11.6.1 of [SV], one has the following analog of l.c Corollary 11.6.2. The proof requires the criteria for discrete series of symmetric spaces due to Kato and Takano [KT2]:

**Proposition 5** *Let  $L^2(X)_P$  the image of  $L^2(X_P)_{disc}$  under  $i_P := i_{G,P}$ . Then one has:*

$$L^2(X) = \sum_{P \in \mathcal{P}_{st}} L^2(X)_P.$$

## 7.5 Scattering theory

From Lemma 14, one proves the analogous of Proposition 13.2.1 in [SV] in which we use  $A_M$  and  $A_L$  instead of  $A_{X,\Theta}$  and  $A_{X,\Omega}$  and where  $P = MU$ ,  $Q = LV$  are  $\sigma$ -parabolic subgroups of  $G$ . This is a step for the analogous of Proposition 13.3.1 in l.c. . We will only recall part (2) of it.

**Proposition 6** *Let  $P = MU, Q = LV$  two standard  $\sigma$ -parabolic subgroups of  $G$ . If the dimensions of  $A_M$  and  $A_L$  are distinct,  $L^2(X)_P$  is orthogonal to  $L^2(X)_Q$ .*

Let  $\Theta_P$  (resp.,  $\Theta_Q$ ) be the set of elements of  $\Sigma(P_\emptyset)$  which are trivial on  $A_M$  (resp.  $A_L$ ). We define  $W(P, Q)$  as the set of elements of  $w \in W(A_\emptyset)$  such that  $w(\Theta_P) = \Theta_Q$ . In particular if  $w \in W(A_M, A_L)$ , it induces an isomorphism between  $A_M$  and  $A_L$ . If  $W(P, Q)$  is non trivial we say that  $P$  and  $Q$  are  $\sigma$ -associated. Let  $c(P) = \sum_{Q \in \mathcal{P}} \text{Card } W(P, Q)$ .

The proof of the analog of l.c. Theorem 14.3.1 (Tiling property of scattering morphisms) is entirely similar. Then one proves the following theorem like Theorem 7.3.1 of l.c. is proved in section 14 of l.c.. Notice that one needs for this proof to establish part of this Theorem for spaces  $X_P$ , but this works like for  $X$ . We recall that  $i_P$  is the map  $i_{G,P}$ .

**Theorem 6** (*Scattering Theorem*) Let  $P = MU$ ,  $Q = LV$ ,  $R$  three standard  $\sigma$ -parabolic subgroups of  $G$ .

(i) If  $P$  and  $Q$  are not  $\sigma$ -associated,  $(i_Q)^t \circ i_P = 0$ .

(ii) If  $P$  and  $Q$  are  $\sigma$ -associated, there exist  $A_M \times G$ -equivariant isometries

$$S_w : L^2(X_P) \rightarrow L^2(X_Q), w \in W(P, Q)$$

where  $A_M$  acts on  $L^2(X_Q)$  via the isomorphism  $A_M \rightarrow A_L$  induced by  $w$ , with the following properties:

$$i_Q \circ S_w = i_P,$$

$$S_{w'} \circ S_w = S_{w'w}, w \in W(P, Q), w' \in W(Q, R),$$

$$(i_Q)^t \circ i_P = \sum_{w \in W(P, Q)} S_w.$$

Let us denote by  $(i_P)^t_{disc}$  the composition of  $(i_P)^t$  with the orthogonal projection to the discrete spectrum. Finally the map

$$\sum_{P \in \mathcal{P}} \frac{(i_P)^t_{disc}}{c(P)^{1/2}} : L^2(X) \rightarrow \bigoplus_{P \in \mathcal{P}} L^2(X_P)_{disc}$$

is an isometric isomorphism onto the subspaces of vectors  $(f_P)_{P \in \mathcal{P}} \in \bigoplus_{P \in \mathcal{P}} L^2(X_P)_{disc}$  satisfying:

$$S_w f_P = f_Q, w \in W(P, Q).$$

In the next section we will explicit the maps  $i_P$ .

## 8 Explicit Plancherel formula

### 8.1 Injectivity of the map $\mathfrak{a}'/W(A) \rightarrow \tilde{\mathfrak{a}}'/W(\tilde{A})$

**Lemma 15** (i) Let  $A$  be a maximal  $\sigma$ -split torus and let  $\tilde{A}$  be a maximal split torus containing  $A$ . It is  $\sigma$ -stable (cf. [HH], Lemma 1.9).

(ii) The set of non zero weights of  $A$  (resp.,  $\tilde{A}$ ) in the Lie algebra of  $G$  is a root system  $\Delta(A)$  ( resp.,  $\Delta(\tilde{A})$ ) which appears as a subset of  $\mathfrak{a}'$  (resp.,  $\tilde{\mathfrak{a}}'$ ).

The set  $\Delta(A)$  is equal to the set of non zero restrictions of elements  $\Delta(\tilde{A})$ .

(iii) Let  $W(A)$  (resp.  $W(\tilde{A})$ ) be the quotient of the normalizer of  $A$  (resp.,  $\tilde{A}$ ),  $N_G(A)$  (resp.  $N_G(\tilde{A})$ ), by its centralizer,  $C_G(A)$  (resp.,  $C_G(\tilde{A})$ ).

Then  $W(A)$  (resp.,  $W(\tilde{A})$ ) identifies with the Weyl group of  $\Delta(A)$  (resp.,  $\Delta(\tilde{A})$ ) and is the set of restrictions to  $\mathfrak{a}$  of the elements of  $W(\tilde{A})$  which normalizes  $\mathfrak{a}$ .

(iv) Let  $\mu, \nu \in \mathfrak{a}'$  which are conjugate by an element of  $W(\tilde{A})$ , then they are conjugate by an element of  $W(A)$ .

*Proof :*

(i) follows from [HH], Lemma 2.4.

(ii) and (iii) follows from [HW], Propositions 5.3 and 5.9.

(iv) It is clear that one may replace  $\mu$  and  $\nu$  by a conjugate element by  $W(A)$ . Hence one

may assume that  $\mu$  and  $\nu$  are dominant for some choice of a set positive roots of  $\Delta(A)$ ,  $\Delta^+(A)$ . Then we choose a set of positive roots for  $\Delta^+(\tilde{A})$  whose non zero restrictions are precisely the elements of  $\Delta^+(A)$ . Hence  $\mu$  and  $\nu$  are dominant for  $\Delta^+(\tilde{A})$  and conjugate by an element of  $W(\tilde{A})$ . Hence they are equal, which proves (iv).  $\square$

**Remark 3** *It follows from (iv) of the previous lemma that the map  $\mathfrak{a}'/W(A) \rightarrow \tilde{\mathfrak{a}}'/W(\tilde{A})$  is injective. This allows to apply the analog of Lemma 14.2.2 of [SV].*

## 8.2 Coinvariants

Let  $P = MU$  be a semistandard  $\sigma$ -parabolic subgroup of  $G$ . Let us prove:

Using our  $G$ -invariant measure on  $X_P$ , the smooth dual of  $C_c^\infty(X_P)$  is isomorphic to  $C^\infty(X_P)$ . (8.1)

An element of the smooth dual of  $C_c^\infty(X_P)$  is fixed by some compact open subgroup  $J$  of  $G$  and is the composition of the  $J$ -average with a linear form on the space of  $J$ -fixed elements of  $C_c^\infty(X_P)$ . A basis of this later space is given by the characteristic functions of  $J$ -cosets. Hence a linear form on this space is given by integration of a  $J$ -fixed element of  $C^\infty(X_P)$ . This proves (8.1).

Similarly one has:

Using our choice of an  $M$ -invariant measure on  $X_M$  (cf. (4.6)), we will identify the smooth dual of  $C_c^\infty(X_M)$  with  $C^\infty(X_M)$ . This identification depends on our choice of  $\mathcal{X}_M^G$ . (8.2)

Let  $(\pi, V)$  be a smooth representation of  $G$  of finite length. Let us define the space of coinvariants:

$$C_c^\infty(X_P)_\pi := \text{Hom}_{\mathbb{C}}(\text{Hom}_G(C_c^\infty(X_P), \pi), \pi). \quad (8.3)$$

As  $\text{Hom}_G(C_c^\infty(X_P), \pi)$  is finite dimensional (cf. (7.1)), one has:

$$\text{Hom}_G(C_c^\infty(X_P)_\pi, \pi) = \text{Hom}_G(C_c^\infty(X_P), \pi).$$

**Definition 3** *If  $\pi$  is a smooth admissible representation of  $G$ , there is a canonical projection*

$$C_c^\infty(X_P) \rightarrow C_c^\infty(X_P)_\pi \rightarrow 0.$$

*If  $\pi = i_{P-}^G \delta$ , we denote this map  $i_{P,\delta}^t$*

The canonical map from  $C_c^\infty(X_P)$  to  $C_c^\infty(X_P)_\pi$  is defined as follows. If  $f \in C_c^\infty(X_P)$ , one defines  $\phi \in C_c^\infty(X_P)_\pi$  by associating to each  $T \in \text{Hom}(C_c^\infty(X_P), \pi)$ , the element  $\phi(T) := T(f)$  of the space of  $\pi$ . It is easy to see that this map is surjective.

Let  $(\delta, E)$  be a unitary irreducible smooth representation of  $M$ . Let  $T \in \text{Hom}_M(C_c^\infty(X_M), \delta)$ . Due to (8.2), the transpose map  $T^t$  might be viewed as an element  $T^t$  of  $\text{Hom}(\delta, C^\infty(X_P))$ . Let us define  $\eta_T = (\eta_{T,x})_{x \in \mathcal{X}_M^G} \in \mathcal{V}(\delta, H)$  ( cf. (5.2) for the notation) by:

$$\eta_{T,x}(\check{e}) := \tilde{T}^t(\check{e})(x), \check{e} \in \check{E}. \quad (8.4)$$

One defines  $Hom_M(C_c^\infty(X_M), \delta)^{disc}$  as the space of  $T \in Hom_M(C_c^\infty(X_M), \delta)$  such that the image of  $\tilde{T}^t$  is a discrete series for  $X_M$ . Let us define:

$$C_c^\infty(X_P)_\delta := (Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes i_{P-}^G \delta. \quad (8.5)$$

$$C_c^\infty(X_P)_\delta[\delta] = (Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes \delta. \quad (8.6)$$

Hence we have:

$$C_c^\infty(X_P)_\delta = i_{P-}^G C_c^\infty(X_P)_\delta[\delta]. \quad (8.7)$$

It can be viewed as a quotient of  $C_c^\infty(X_P)$  as follows (cf. [SV] before Equation (15.12)). From the Lemma 12, one has an injective map defined by induction:

$$0 \rightarrow Hom_M(C_c^\infty(X_M), \delta)_{disc} \rightarrow Hom_G(C_c^\infty(X_P), i_{P-}^G \delta).$$

Hence, using the transpose map and taking into account the notation (8.3) one has a surjective map:

$$C_c^\infty(X_P)_{i_{P-}^G \delta} = Hom_G(C_c^\infty(X_P), i_{P-}^G \delta)' \otimes i_{P-}^G \delta \rightarrow C_c^\infty(X_P)_\delta \rightarrow 0.$$

Together with Definition 3, this shows that

$$C_c^\infty(X_P)_\delta \text{ is a quotient of } C_c^\infty(X_P). \quad (8.8)$$

The smooth dual of  $C_c^\infty(X_P)_\delta$  is denoted  $C^\infty(X_P)^\check{\delta}$  and one has

$$C^\infty(X_P)^\check{\delta} = Hom_M(C_c^\infty(X_M), \delta)^{disc} \otimes i_{P-}^G \check{\delta}.$$

From (8.8) it can be viewed as a subspace of  $C^\infty(X_P)$ .

### 8.3 Eisenstein integral maps and their transpose

**Definition 4** We use the fact that the Eisenstein integral associated to  $\delta_\chi$  are well defined for  $\chi$  in the complementary set of the zero set of a non zero polynomial function on  $X(M)_\sigma$ . For such a  $\chi$ , we define a map called Eisenstein integral map:

$$E_{P, \delta_\chi} \in Hom_G(Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc} \otimes i_P^G \delta_\chi, C^\infty(X)).$$

by

$$E_{P, \delta_\chi}(T \otimes v) = E(P, \delta_\chi, \eta_T, v), T \in Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc}, v \in i_P^G \delta_\chi.$$

We keep the notation of the preceding subsection. Let us denote by  $ev_1$  the map

$$ev_1 : (Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc})' \otimes i_P^G \check{\delta}_\chi \rightarrow (Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc})' \otimes \check{E}$$

defined by:

$$ev_1(\theta \otimes v) = \theta \otimes v(1), \theta \in (Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc})', v \in i_P^G \check{\delta}_\chi.$$

If  $\phi \in C_c^\infty(X_M)$ , let  $q_\delta(\phi)$  be the canonical element of  $(\text{Hom}_M(C_c^\infty(X_M), \check{\delta})^{disc})' \otimes \check{E}$  defined as follows. The later space appears as the smooth dual of  $\text{Hom}_M(C_c^\infty(X_M), \check{\delta})^{disc} \otimes E$  and we define

$$\langle q_\delta(\phi), T \otimes e \rangle := \langle e, T(\phi) \rangle, e \in E, T \in \text{Hom}_M(C_c^\infty(X_M), \check{\delta})^{disc}.$$

Identifying the smooth dual of  $C_c^\infty(X_M)$  to  $C^\infty(X_M)$  (cf. (8.2)), one has also:

$$\langle q_\delta(\phi), T \otimes e \rangle = \langle \tilde{T}^t(e), \phi \rangle. \quad (8.9)$$

Let us denote, by abuse of notation, the restriction of the transpose map of  $E_{P, \delta_\chi}$  to  $C_c^\infty(X)$  by  $E_{P, \delta_\chi}^t$ .

**Lemma 16** *One has*

$$E_{P, \delta_\chi}^t \in \text{Hom}_G(C_c^\infty(X), (\text{Hom}_M(C_c^\infty(X_M), \check{\delta}_\chi)^{disc})' \otimes i_P^G \check{\delta}_\chi)$$

and

$$ev_1((E_{P, \delta_\chi}^t(f))) = q_\delta(f^P), f \in C_c^\infty(X).$$

*Proof :*

Let  $e \in E$ ,  $T \in \text{Hom}_M(C_c^\infty(X_M), \check{\delta})^{disc}$ . Let  $J$  be a compact open subgroup of  $G$  with a  $\sigma$ -factorization for  $P$  and such that  $J_M$  fixes  $e$  and  $f$  and let  $v_\chi := v_{e, \delta_\chi}^{P, J}$  the element of  $i_P^G \check{\delta}_\chi$  which is invariant by  $J$ , whose support is equal to  $PJ$  and whose value at 1 is equal to  $e$  (for the existence see e.g. [CD] Equation (3.2)). Notice that, from (4.8), one has:

$$v_\chi \text{ has its support equal to } PJ_{U^-} = PJ_H \subset PH \quad (8.10)$$

We will compute in two ways:

$$I := \langle E_{P, \delta_\chi}^t(f), T \otimes v_\chi \rangle$$

We take into account the expression of the duality of  $i_P^G \delta$  and  $i_P^G \check{\delta}$  (cf. (5.1) and (8.10)). This leads to our first expression of  $I$ :

$$I = \text{vol}(J_{U^-}) \langle ev_1(E_{P, \delta_\chi}^t(f)), T \otimes e \rangle \quad (8.11)$$

In order to compute  $I$  in an other way we use a transposition:

$$I = \int_{H \backslash G} f(\dot{g}) E_{P, \delta_\chi}(T \otimes v_\chi)(\dot{g}) d\dot{g}.$$

For  $Re\chi$  sufficiently  $P$ -dominant, one has from (5.4) and the definition of  $\eta_T$  (cf.(8.4)):

$$I = \int_{H \backslash G} f(\dot{g}) \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1} \cdot H \backslash x^{-1} \cdot H} \tilde{T}^t(v_\chi(yx^{-1}g))(x) dy d\dot{g}.$$

One makes the change of variable  $g' = x^{-1}.g$  and then the Fubini theorem that one can use because  $f$  is compactly supported. One gets:

$$I = \sum_{x \in \mathcal{X}_M^G} \int_{(M \cap x^{-1}.H) \backslash G} f(x.g) \tilde{T}^t(v_\chi(gx^{-1}))(x) d\dot{g}.$$

We make the change of variable  $g'' = gx^{-1}$ . We use the integration formula (4.1) and our choice of measure on  $M \cap x^{-1}.H \backslash M$ . As  $v_\chi$  has its support in  $PJ_{U-}$  and  $f$  and  $v_\chi$  are  $J$ -invariant, one gets:

$$I = \text{vol}(J_{U-}) \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1}.H \backslash M} \delta_P(m^{-1}) \int_U f(xum) du \tilde{T}^t(v_\chi(m))(x) dm.$$

But the change variable  $u' = m^{-1}um$  shows that:

$$I = \text{vol}(J_{U-}) \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1}.H \backslash M} \int_U f(xmu) du \tilde{T}^t(v_\chi(m))(x) dm.$$

From the intertwining property of  $T$  one has:

$$\tilde{T}^t(v_\chi(m))(x) = \delta_P^{1/2}(m) \tilde{T}^t(e)(xm).$$

With our choices of measures one deduces:

$$I = \text{vol}(J_{U-}) \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1}.H \backslash M} f^P(\dot{x}m) \tilde{T}^t(e)(\dot{x}m) dm$$

In other words

$$I = \text{vol}(J_{U-}) \langle f^P, \tilde{T}^t(e) \rangle,$$

and (8.9) implies:

$$I = \text{vol}(J_{U-}) \langle q_\delta(f^P), T \otimes e \rangle.$$

From (8.11) and Lemma 16 (i) one deduces the equality:

$$ev_1((E_{P, \delta_\chi})^t(f)) = q_\delta(f^P).$$

□

## 8.4 Canonical quotient and the small Mackey restriction

We follow the terminology of [SV], section 15. Let  $\tau$  be a finite length smooth representation of  $M$ . If the intertwining integral:

$$A(P, P^-, \tau) : i_{P^-}^G \tau \rightarrow i_P^G \tau$$

is well defined, the *canonical quotient* is the composition:

$$(i_{P^-}^G \tau)_P \xrightarrow{j_P(A(P, P^-, \tau))} (i_P^G \tau)_P \rightarrow \tau$$



where the right map is the evaluation at 1 (cf. [SV] Equation (15.8)). If  $\tau = C_c^\infty(X_P)_\delta[\delta]$ , the canonical quotient in this case is denoted  $c_\delta$  and taking into account (8.7) one has:

$$c_\delta : (C_c^\infty(X_P)_\delta)_P \rightarrow C_c^\infty(X_P)_\delta[\delta].$$

Let  $(\pi, V)$  be a smooth representation of  $G$ . The Mackey restriction (cf. [SV] section 15.4.3) is the map

$$Mack : Hom_G(C_c^\infty(X), \pi) \rightarrow Hom_M(C_c^\infty(X_M), \pi_P)$$

obtained by taking the Jacquet functor to any element  $T$  of  $Hom_G(C_c^\infty(X), \pi)$ , and restricting it to  $C_c^\infty(X_M)$  which is identified by  $m_P^X$  (cf Lemma 16 (iii)) with a subspace of the normalized Jacquet module of  $C_c^\infty(X)$ .

If  $\pi = i_{P^-}^G \tau$ , and the intertwining integral  $A : i_{P^-}^G \tau \rightarrow i_P^G \tau$  is bijective the small Mackey restriction is the composition of the canonical quotient with the Mackey restriction  $Mack$ :

$$sMack : Hom_G(C_c^\infty(X), \pi) \rightarrow Hom_M(C_c^\infty(X_M), \tau)$$

If  $\pi = C_c^\infty(X_P)_\delta$ , we denote by  $sMack_\delta$  the small Mackey restriction. If  $T \in Hom_G(C_c^\infty(X), \pi)$ ,

$$sMack_\delta(T) \in Hom_M(C_c^\infty(X_M), Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes \delta$$

## 8.5 Normalized Eisenstein integrals

**Definition 5** Let  $P$  be a semistandard  $\sigma$ -parabolic subgroup of  $G$ . We define the normalized integral

$$E_{P, \delta_\chi}^0 \in Hom_G(Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc} \otimes i_{P^-}^G \delta_\chi, C_c^\infty(X))$$

by:

$$E_{P, \delta_\chi}^0 := E_{P, \delta_\chi} \circ (Id \otimes A(P^-, P, \delta_\chi)^{-1})$$

which is rational in  $\chi \in X(M)_\sigma$ .

By the formula of the transpose of intertwining integrals (cf. [W] IV.1(11) and denoting by  $(E_{P, \delta_\chi}^0)^t$  the restriction of the transpose of  $E_{P, \delta_\chi}^0$  to  $C_c^\infty(X)$ , one has

$$(E_{P, \delta_\chi}^0)^t = (Id \otimes A(P, P^-, \delta_\chi)^{-1}) \circ (E_{P, \delta_\chi})^t.$$

From this it follows

$$sMack((E_{P, \delta_\chi}^0)^t) \in Hom(C_c^\infty(X_M), Hom_M(C_c^\infty(X_M), \delta_\chi)^{disc})' \otimes \delta_\chi$$

is equal to

$$ev_{1j_P}(A(P, P^-, \delta) \circ (E_{P, \delta}^0)^t).$$

From Lemma 16, one deduces:

$$\text{The map } sMack((E_{P, \delta_\chi}^0)^t) \text{ is equal to the map } q_{\delta_\chi}. \quad (8.12)$$

Our definition of normalized Eisenstein integrals differs from the one in [SV], Equation (15.30) for  $G$ -split and  $X$  spherical. Here we do not use the Radon transform, but we use that the opposite of a  $\sigma$ -parabolic subgroup is a  $\sigma$ -parabolic subgroup. From (8.12), our Eisenstein integrals maps have the same small Mackey restrictions than the ones defined in l.c. (cf. (15.36)). (8.13)

## 8.6 Explicit Plancherel formula

Let

$$L^2(X_M)_{disc} = \int_{\hat{M}}^{\oplus} \check{I}_{\delta} d\nu(\delta)$$

where  $\check{I}$  is a unitary representation of  $M$  isomorphic to a direct sum of copies of  $\delta$ . From Lemma 13, one has

$$L^2(X_P)_{disc} = \int_{\hat{M}}^{\oplus} \check{H}_{\delta} d\nu(\delta)$$

where  $\check{H}_{\delta}$  is the unitarily induced representation from  $P^-$  to  $G$  of  $\check{I}_{\delta}$ . Let  $\check{H}_{\delta}^{\infty}$  be its space of smooth vectors. With the notation of (8.5), its space of smooth vectors is equal to  $C^{\infty}(X_P)^{\delta}$ .

Let  $f \in C_c^{\infty}(X_P)$  and let us write its discrete component

$$f_{disc} = \int_{\hat{M}} f^{\delta} d\nu_{disc}(\delta),$$

where  $f^{\delta} \in C^{\infty}(X_P)^{\delta}$ . Its image by the Bernstein morphism  $i_P(f)$  satisfies:

$$i_P(f_{disc}) = \int_{\hat{M}} i_{P,\delta}(f) d\nu(\delta).$$

for some maps  $i_{P,\delta} : \check{H}_{\delta}^{\infty} \rightarrow C^{\infty}(X)$  defined for almost all  $\delta$  (cf. [SV] Equation (15.6)).

One has the analog of Lemma 15.4.4 of [SV]. As the analogous of section 15.6 of [SV] is identical, together with (8.12), this leads to the analog of Th 15.5.5 in [SV]:

**Proposition 7** *The small Mackey restriction of  $sMack((E_{P,\delta}^0)^t)$  and  $i_{P,\delta_{\chi}}^t$  are equal for almost all  $\chi \in X(M)_{\sigma,u}$ .*

Also by the uniqueness result of [BD] recalled in (5.3), for almost all  $\chi \in X(M)_{\sigma,u}$ , every element  $F$  of  $Hom_G(C_c^{\infty}(X), i_{P^-}^G \delta_{\chi})$  is given in term of the normalized Eisenstein integral i.e. is of the form

$$F = E(P, \delta_{\chi}, \eta_T, v) \circ A(P^-, P, \delta_{\chi})^{-1}$$

for a unique  $T \in Hom_M(C_c^{\infty}(X_M), \check{\delta}_{\chi})$ . Using (8.12) or rather its immediate generalization by replacing  $Hom_M(C_c^{\infty}(X_M), \check{\delta}_{\chi})^{disc}$  by  $Hom_M(C_c^{\infty}(X_M), \check{\delta}_{\chi})$  one sees that the small Mackey restriction of  $F$  is equal to  $T$ . Hence one has:

**Proposition 8** *The small Mackey restriction*

$$sMack : Hom_G(C_c^\infty(X), i_{P-}^G \delta_\chi) \rightarrow Hom_M(C_c^\infty(X_M), \delta_\chi)$$

*is injective for almost all  $\chi \in X(M)_{\sigma, u}$ .*

**Corollary 2** *For almost all  $\chi$  in  $X(M)_{\sigma, u}$ , one has:*

$$i_{P, \delta_\chi} = E_{P, \delta_\chi}^0.$$

**Theorem 7** *Let  $f \in C_c^\infty(X_P)$  and let us write its discrete component*

$$f_{disc} = \int_{\hat{M}}^{\oplus} f^\delta d\nu_{disc}(\delta),$$

*where  $f^\delta \in C^\infty(X_P)^\delta$ .*

*Its image by the Bernstein morphism  $i_P(f)$  satisfies:*

$$i_P(f)(x) = \int_{\hat{M}} E_{P, \delta}^0(f^\delta)(x) d\nu(\delta), x \in X.$$

In combination with the scattering theorem (cf. Theorem 6), one deduces:

**Theorem 8** *The norm on  $L^2(X)_P$ ,  $\|\cdot\|_P$ , admits the decomposition:*

$$\|\Phi\|_P^2 = \frac{4}{Card(W(A_P, A_P))} \int_{\hat{M}} \|E_{P, \delta}^{0t}(\Phi)\|_\delta^2 d\nu(\delta),$$

*where the measure and norms on the right hand side of the equality are the discrete part of the Plancherel decomposition of  $L^2(X_P)$ .*

## 9 Appendix: Rational representations

In this section we establish some results on rational representations of  $G$  which are needed to extend the results of [L] and [BD], which are established when  $\mathbf{F}$  is of characteristic zero, to the case where  $\mathbf{F}$  is simply of characteristic different from 2.

### 9.1 Rational representations and parabolic subgroups

Let  $\underline{G}$  be a reductive algebraic group defined over a non archimedean local field  $\mathbf{F}$ , whose group of  $\mathbf{F}$ -points is equal to  $G$ . We will use similar notations for the subgroups of  $G$ .

Let  $A_0$  be a maximal split torus of  $G$  and let  $P_0$  be a minimal parabolic subgroup of  $G$ . Let  $P$  be a parabolic subgroup of  $G$  which contains  $P_0$ . Let  $T$  be a maximal  $\mathbf{F}$ -torus of  $\underline{G}$  which contains  $\underline{A}_0$ . Let  $B$  be a Borel subgroup of  $\underline{G}$ , which contains  $T$  and contained in the opposite parabolic subgroup to  $P_0$  which contains  $A_0, \underline{P}_0^-$ . One denotes by  $\Sigma(T)$  the set of roots of  $T$  in the Lie algebra of  $\underline{G}$ . One denotes by  $\Lambda(T)$  (resp.,  $\Lambda(T)_{rac}$ ) the weight lattice (resp., the root lattice) of  $T$  with respect to  $\underline{G}$ . We adopt similar notations for  $A_0$ . Let  $\Gamma$  be the absolute Galois group of  $\mathbf{F}$  which acts on these lattices. Let  $\Lambda^+(T)$  be the set of dominant weights for  $T$  relative to  $B$ . Let  $\Lambda^+(A_0)$  (resp.,  $\Lambda^+(A_0)_{rac}$ ) the set of dominant elements for  $P_0^-$  of  $\Lambda^+(A_0)$  (resp.,  $\Lambda(A_0)_{rac}$ ).

**Definition 6** One denotes by  $\Lambda_M^+(T)$  the set of elements  $\lambda$  of  $\Lambda^+(T)$  such that  $G$  has a rational finite dimensional irreducible representation, defined over  $\mathbf{F}$ , with highest weight  $\lambda$  relative to  $B$ ,  $(\pi_\lambda, V_\lambda)$ , with the following property:

$$\text{Any non zero vector of weight } \lambda \text{ under } T, v_\lambda, \text{ transforms under } M \text{ by a rational character of } M, \text{ denoted } \Lambda. \quad (9.1)$$

The subset of  $\lambda \in \Lambda_M^+(T)$  which satisfies the following property is denoted by  $\Lambda_M^{++}(T)$ .

$$\text{There exists } v'_\lambda \text{ in the dual } V'_\lambda \text{ of } V_\lambda, \text{ invariant by } U \text{ and such that the coefficient } c_\lambda(g) = |\langle \pi_\lambda(g)v_\lambda, v'_\lambda \rangle|_{\mathbf{F}} \text{ is equal to zero on the complementary subset in } G \text{ of } UMu^-. \quad (9.2)$$

The goal of this subsection is to produce sufficiently many elements of  $\Lambda_M^+(T)$ .

**Proposition 9** (i) Let  $T_{an}$  be the anisotropic component of  $T$ . There exists  $n \in \mathbb{N}^*$  such that any element  $\lambda$  of  $n\Lambda^+(A_0)$  extends uniquely to an element  $\mu$  of  $\Lambda(T)_{rac}$  trivial on  $T_{an}$ .

(ii) If  $\lambda$  is orthogonal to the simple roots of  $A_0$  in the Lie algebra of  $U_0^- \cap M$  then  $\mu$  is element of  $\Lambda_M^+(T)$ .

(iii) If moreover  $\lambda$  is not orthogonal to the other simple roots of  $A_0$  in the Lie algebra of  $U_0^-$ ,  $\mu$  is an element of  $\Lambda_M^{++}(T)$ .

For the proof we need several lemmas.

Let  $\beta$  be an element of the set,  $\Sigma(A_0)$ , of roots of  $A_0$  in the Lie algebra of  $G$ . One defines:

$$\underline{\beta} := \sum_{\alpha \in \Sigma(T), \alpha|_{A_0} = \beta} \alpha.$$

One sees easily that:

$$\text{There exists } n' \in \mathbb{N}^* \text{ such that, for all } \beta \in \Sigma(A_0), \text{ there exists } n'_\beta \in \mathbb{N}^* \text{ such that } n'_\beta \underline{\beta}|_{A_0} = n' \beta. \quad (9.3)$$

We fix, once for all, such integers  $n'$  and  $n'_\beta$

**Lemma 17** Every element  $\lambda$  of  $n'\Lambda_{rac}(A_0)$  extends uniquely to an element  $\mu$  of  $\Lambda_{rac}(T)$  trivial on the anisotropic component  $T_{an}$  of  $T$ , invariant by  $\Gamma$  and by  $W(\underline{M}_0, T)$ .

Let us denote by  $(n'\Lambda_{rac}(A_0))^\sim$  the lattice generated by the  $n'_\beta \underline{\beta}$ ,  $\beta \in \Sigma(A_0)$ . From their definition, one sees that the elements of  $(n'\Lambda_{rac}(A_0))^\sim$  are invariant under  $\Gamma$  and are elements of  $\Lambda_{rac}(T)$ . One remarks that every element  $\mu$  of  $(n'\Lambda_{rac}(A_0))^\sim$  is invariant by the Weyl group of  $\underline{M}_0$  relative to  $T$ ,  $W(\underline{M}_0, T)$ .

Let us show any element  $\mu$  is trivial on  $T_{an}$ . One can choose  $T$  such that it contains a maximal torus defined over  $\mathbf{F}$ ,  $T_1$ , of the derived group of  $\underline{M}_0$ . Actually, by conjugacy, one sees that any  $T$  has this property. Moreover  $T$  contains the maximal anisotropic torus  $C_{an}$  of the center of  $M_0$ . The product  $T_1 C_{an} \underline{A}_0$  is a torus. For reasons of dimension it is a maximal torus of  $G$ . Hence  $T = T_1 C_{an} \underline{A}_0$ . Notice that  $T_1 C_{an}$  is the anisotropic

component  $T_{an}$  of  $T$ . As  $\mu$  is  $W(\underline{M}_0, T)$ -invariant, the restriction of  $\mu$  to  $T_1$  is trivial. As  $C_{an}$  is anisotropic, the invariance by  $\Gamma$  of  $\mu$  shows that its restriction to  $C_{an}$  is trivial. This proves the existence part of the Lemma. As  $T = T_{an}\underline{A}_0$  the unicity follows.  $\square$

**Lemma 18** (i) *There exists  $n \in \mathbb{N}$  such that  $n\Lambda(A_0) \subset n'\Lambda_{rac}(A_0)$ .*  
(ii) *If  $\lambda \in n\Lambda^+(A_0)$ , its extension  $\mu$  to  $T$  given by the preceding lemma is the highest weight of a rational representation of  $G$ , defined over  $\mathbf{F}$ , denoted  $(\pi_\mu, V_\mu)$ .*

*Proof :*

(i) The lattice  $n'\Lambda_{rac}(A_0)$  is contained in the lattice  $\Lambda(A_0)$ . As these lattices are of the same rank, there exists  $n \in \mathbb{N}^*$  such that  $n\Lambda(A_0) \subset n'\Lambda_{rac}(A_0)$ .  
(ii) From (i) and the preceding lemma, if  $\lambda \in n\Lambda^+(A_0)$ ,  $\mu$  is in  $\Lambda_{rac}(T) \subset \Lambda(T)$ . Moreover if  $\alpha$  is a root of  $T$  in the Lie algebra of  $\underline{G}$ ,  $\langle \mu, \alpha \rangle = \langle \lambda, \alpha|_{A_0} \rangle$ . Hence  $\mu$  is a dominant weight. From the preceding Lemma, it is invariant by  $\Gamma$ . Then [T], Theorem 3.3 and Lemma 3.2 implies (ii).  $\square$

**Lemma 19** *Let  $\lambda \in n\Lambda^+(A_0)$  and  $\mu$  as in Lemma 17. Then, with the notation of the preceding lemma,  $M_0$  acts on a non zero highest weight vector of  $(\pi_\mu, V_\mu)$  by a rational character of  $M_0$  again denoted by  $\mu$ .*

*Proof :*

As  $\pi_\mu$  is defined over  $\mathbf{F}$ , it is enough to prove that  $v_\mu$  transforms under a rational character of  $M_0$ . In order to prove this, one can work with the algebraic closure. The invariance of  $\mu$  by  $W(\underline{M}_0, T)$  (cf. Lemma 17), the fact that the space of weight  $\mu$  in  $V_\mu$  is of dimension one (cf. [Hu], Proposition 33.2) together with the Bruhat decomposition of  $\underline{M}_0$  allow to prove the Lemma.  $\square$

*Proof of Proposition 9*

Let  $\lambda \in n\Lambda^+(A_0)$  be as in the statement of Proposition 9 (ii) i.e.  $\lambda$  is orthogonal to the simple roots of  $A_0$  in the Lie algebra of  $U_0^- \cap M$ . Let  $\mu$  be as in Lemma 17. Let  $(\pi_\mu, V_\mu)$  be as in Lemma 18, and let  $v_\mu$  be a non zero highest weight vector. One has to prove that  $v_\mu$  transforms under  $M$  by a rational character of  $M$  that we will still denote by  $\mu$ . It is enough to prove that the line  $\mathbf{F}\mu$  is stable by the action of  $M$ . One shows, using the preceding Lemma, by a proof analogous to the one of [Hu], Proposition 31.2 and using the density of  $U_0^- M_0 U_0$  in  $G$ , that the  $A_0$ -weight space of  $V_\mu$  for the weight  $\lambda$  is one dimensional. The Weyl group,  $W(M, A_0)$ , of  $M$  relative to  $A_0$  fixes  $\lambda$  from the hypothesis on  $\lambda$ . One finishes the proof of our assertion on the action  $M$  on  $v_\mu$  by using the Bruhat decomposition of  $M$  relative to  $P_0^- \cap M$ . Hence  $\mu \in \Lambda_M^+(T)$ .

Now we assume moreover that  $\lambda$  is not orthogonal to the other simple roots of  $A_0$  in the Lie algebra of  $U_0^-$ ,  $\mu$  is an element of  $\Lambda_M^{++}(T)$ . One sees like in l.c. that the weights of  $A_0$  in  $V_\mu$  are of the form  $\nu = \lambda + \sum_{\beta \in \Delta(P_0)} c_\beta \beta$ , where for all  $\beta$  in the set of simple roots  $\Delta(P_0)$  of  $A_0$  in the Lie algebra of  $U_0$ ,  $c_\beta \in \mathbb{N}$  (we recall that  $B$  is contained in  $\underline{P}_0^-$ ). We consider the hyperplane of  $V_\mu$  generated by the  $A_0$ -weight spaces for the weights distinct from  $\lambda$ . From what we have just established on the weights of  $A_0$  in  $V_\mu$

and from [Hu], Proposition 27.2, one sees that this hyperplane is stable by  $U$ . Hence the linear form on  $V_\mu, v'_\mu$ , which vanishes on this hyperplane and which takes the value 1 on  $v_\mu$  transforms under a rational character of the unipotent group  $U$ . This implies that  $v'_\mu$  is invariant by  $U$ .

Let  $c_\mu$  be the real valued function on  $G$  defined by:

$$c_\mu(g) = |\langle \pi_\mu(g)v_\mu, v'_\mu \rangle|_{\mathbf{F}}, g \in G.$$

Let us show that our hypothesis on  $\lambda$  implies that  $c_\mu$  vanishes on the complimentary subset of  $UMU^-$  in  $G$ . From the Bruhat decomposition for  $P_0^-$  an element of this complimentary subset can be written  $g = wmu^-$ , where  $w$  represents an element of  $W(G, A_0)$  which is not in  $W(M, A_0)$ . Then  $c_\mu(g)$  is proportional to  $|\langle \pi_\mu(w)v_\mu, v'_\mu \rangle|_{\mathbf{F}}$ . But  $\pi_\mu(w)v_\mu$  is of weight  $w\lambda$  under  $A_0$ . This weight is distinct of  $\lambda$  as  $\lambda$  is not orthogonal to the simple roots of  $A_0$  in the Lie algebra  $U_0^-$  which are not roots of  $A_0$  in  $M$ . This implies that  $|\langle \pi_\mu(w)v_\mu, v'_\mu \rangle|_{\mathbf{F}}$  is equal to zero. Hence  $c_\mu(g)$  is equal to zero as wanted. This achieves the proof of the Proposition.  $\square$

## 9.2 $H$ -distinguished rational representations of $G$

Proposition 9 allows to extend the results of [BD] section 2.7 and especially Propositions 2.9, 2.11 to a non archimedean local field,  $\mathbf{F}$ , of characteristic different from 2. Let  $\Sigma(G, A_0)$  (resp.,  $\Sigma(P_0, A_0)$  or  $\Sigma(P_0)$ ) the set of roots of  $A_0$  in the Lie algebra of  $G$  (resp.,  $P_0$ ). We denote by  $\Delta(P_0)$  the set of simple roots of  $\Sigma(P_0)$ .

Let  $P = MU$  be a standard  $\sigma$ -parabolic subgroups of  $G$ . We will use the notation of the main body of the article. Let us assume that  $A_\emptyset \subset A_0$ , which is automatically  $\sigma$ -stable, and  $P_0 \subset P_\emptyset$ . Let  $\{\alpha_1, \dots, \alpha_{m_0}\}$  be the simple roots of  $\Sigma(P_0)$  written in such a way that  $\{\alpha_1, \dots, \alpha_{m_\emptyset}\}$  are the simple roots in the Lie algebra of  $U_\emptyset$ ,  $\{\alpha_1, \dots, \alpha_m\}$  are the simple roots in the Lie algebra of  $U$ . One has the fundamental weights of  $\Sigma(P_0, A_0)$ ,  $\delta_1, \dots, \delta_l$ .

Let  $i = 1, \dots, m$  and  $\lambda_i = n\delta_i$  with  $n$  as in Proposition 9. From this proposition, there exists a unique rational character of  $T$ ,  $\mu$ , trivial on  $T_{an}$  and whose restriction to  $A_0$  is equal to  $\lambda_i$  and such that  $\lambda_i \in \Lambda_M^+(T)$  and  $\mu$  is the highest weight of an irreducible finite dimensional rational representation of  $G$ , denoted by  $(\pi_\mu, V_\mu)$ . Moreover if  $v_\mu$  is a non zero highest weight vector in  $V_\mu$ , the space  $\mathbf{F}v_\mu$  is  $P$ -invariant. We denote again by  $\mu$  the rational character of  $M$  which describes the action of  $M$  on  $v_\mu$ . One denotes by  $v'_\mu$  the unique element of  $V'_\mu$  of weight  $\mu^{-1}$  under  $M$  and such that  $\langle v'_\mu, v_\mu \rangle = 1$ .

Let  $\nu := \mu(\mu^{-1} \circ \sigma) \in \Lambda(T)$  and let  $(\tilde{\pi}_\nu, \tilde{V}_\nu)$  be the rational representation of  $G$   $(\pi_\mu \otimes (\pi'_\mu \circ \sigma), V_\mu \otimes V'_\mu)$ . Let  $\tilde{v}_\nu := v_\mu \otimes v'_\mu$  which is of weight  $\nu$  under the representation  $\tilde{\pi}_\nu$  restricted to  $M$ . Then there exists a non zero  $H$ -invariant vector, under  $\tilde{\pi}_\nu$  in  $\tilde{V}_\nu = (V_\mu \otimes V'_\mu)' \simeq V'_\mu \otimes V_\mu \simeq \text{End}V_\mu$ , namely the identity that we will denote  $e'_{\nu, H}$ . It satisfies  $\langle e'_{\nu, H}, \tilde{v}_\nu \rangle = 1$ .

Let us show that  $\nu = 2\mu$ . As  $\sigma$  preserves  $T_{an}$ , the character  $\mu^{-1} \circ \sigma$  of  $T$  is trivial on  $T_{an}$ . Its restriction to  $A_0$  is equal to  $\lambda$ . From the unicity statement of  $\mu$ , it is equal to  $\mu$ . This proves our claim.

From this it follows that

Proposition 2.9 and 2.11 of [BD] extend to a non archimedean local field,  $\mathbf{F}$ , of characteristic different from 2. This shows that the results of [BD], section 2.8, 2.9 are valid for such a field. Also, the Lemma 1 (resp., section 3.2) in [L] is true also for such a field  $\mathbf{F}$  due to Proposition 2.3 of [CD] (resp., the Proposition 9 of the present article). Hence the results of [L] are valid for such a field  $\mathbf{F}$ . (9.4)

## 10 References

- [AAG] Aizenbud A., Avni N., Gourevitch D., Spherical pairs over close local fields. *Comment. Math. Helv.* 87 (2012) 929–962.
- [BenO] Benoist Y., Oh H., Polar decomposition for  $p$ -adic symmetric spaces. *Int. Math. Res. Not. IMRN* 2007, Art. ID rnm121.
- [Ber] Bernstein J., On the support of Plancherel measure. *J. Geom. Phys.* 5 (1988) 663–710 (1989).
- [BD] Blanc P., Delorme P., Vecteurs distributions  $H$ -invariants de représentations induites pour un espace symétrique réductif  $p$ -adique  $G/H$ , *Ann. Inst. Fourier*, 58 (2008), 213–261.
- [CD] Carmona J., Delorme P., Constant term of  $H$ -forms arXiv:1105.5059
- [D] Delorme P., Constant term of smooth  $H_\psi$ -invariant functions, *Trans. Amer. Math. Soc.* 362 (2010), 933–955.
- [HH] Helminck A.G., Helminck G.F., A class of parabolic  $k$ -subgroups associated with symmetric  $k$ -varieties. *Trans. Amer. Math. Soc.* 350 (1998) 4669–4691.
- [HW] Helminck A. G., Wang S. P., On rationality properties of involutions of reductive groups. *Adv. Math.* 99 (1993) 26–96.
- [Hu] Humphreys J. E, *Linear algebraic groups*, Graduate Text In Math. 21, Springer, 1981.
- [KT1] Kato S., Takano K., Subrepresentation theorem for  $p$ -adic symmetric spaces, *Int. Math. Res. Not. IMRN* 2008, no. 11.
- [KT2] Kato S., Takano K., Square integrability of representations on  $p$ -adic symmetric spaces. *J. Funct. Anal.* 258 (2010) 1427–1451.
- [L] Lagier N., Terme constant de fonctions sur un espace symétrique réductif  $p$ -adique, *J. of Funct. An.*, 254 (2008) 1088–1145.
- [R] Renard D., Représentations des groupes réductifs  $p$ -adiques. *Cours Spécialisés*, 17. Société Mathématique de France, Paris, 2010.
- [T] Tits J., Représentations linéaires d’un groupe réductif sur un corps quelconque, *J. Reine Angew. Math.* 247 1971 196–220.
- [SV] Sakellaridis Y., Venkatesh A., Periods and harmonic analysis on spherical varieties, arXiv:1203.0039
- [W] Waldspurger J.-L., La formule de Plancherel pour les groupes  $p$ -adiques (d’après Harish-Chandra), *J. Inst. Math. Jussieu* 2 (2003), 235–333.

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